Computational experiments on Contested Pile methods for the fair allocation of indivisible items

Rudolf Vetschera[a], D. Marc Kilgour[b]
[a] University of Vienna, Austria
[b] Wilfrid Laurier University, Canada

COST action meeting
Sibiu, Romania
Oct. 20, 2014

Agenda

- **Contested Pile procedures**
- **Design parameters**
- **Undercut procedure**
- **Results**
  - Contested pile
  - Efficiency and fairness
  - Properties of undercut
- **Strategic play**
- **Conclusions**

Problem definition

- **Two players**
- **A finite set of indivisible items**
- **Players rank items differently**
- **Rankings are private information**
- **Task: allocate items to players so that allocation is**
  - efficient
  - fair
  - and procedure is not (easily) manipulable
- **Further assumptions (for computational experiments)**
  - Cardinal evaluation of item i for player j: v_{ij}
  - Preferences are additive: $u_i(S) = \sum_{j \in S} v_{ij}$

Basic contested pile procedure

- **Both player simultaneously claim an (available) item**
- **If different items are claimed, each player receives the desired item**
- **If both claim the same item, that item is put on "contested pile"**
- **In both cases, the item(s) claimed are no longer available**
- **Repeat above steps until all items are assigned to players or placed on contested pile**

Note: at the end of the procedure, all items on contested pile are ranked identically by both players (assuming truthful behavior of players)

Example

Item	u_{1i}	u_{2i}
A	4	1
B	3	3
C	2	4
D	1	2

Round	Player 1 claims	Player 2 claims	Result
1	A	C	Assigned
2	B	B	Contested
3	D	D	Contested

Note: both players prefer B over D

Design parameters

- **Generation phase**
  - Direction: Claim or reject
  - Number of items in each step
- **Splitting phase**
  - Methods to split CP
    - Balanced alteration
    - Borda max-min
    - Undercut

Research questions

- **How does design of generation phase influence**
  - size and composition of contested pile
  - efficiency and fairness of outcomes
- **How does choice of splitting procedure influence efficiency and fairness of outcomes**
- **… in particular, how effective is the undercut procedure**
- **How strongly are different variants affected by strategic behavior?**

Computational experiments

- **Round-robin tournament between different players**
  (sincere and different types of strategic behavior)
- **Each type plays against all types (including copy of itself)**
- **Randomly generated problems**
  pre-specified correlation of utilities
- **For each problem**
  - **all generation methods**
  - **all splitting methods**
- **For evaluation, assume cardinal utilities and additive preferences**

Simulation system

Problem instance:
- Set of items
- Utilities to players
- Allocation status

Problem generator
- Size
- Correlation between utilities

CP Builder:
- BuildCP(prob, p1,p2) method

one step on problem p for players p1 and p2

CP Splitter:
- SplitCP(prob) method

calculates utilities for both players after applying splitting method

Optimal (max-min) split as benchmark

Players:
- Select(n) method
- Reject(n) method

according to strategy

Hypotheses

- **Direction should have no effect**
  both directions provide basically the same information
- **Claiming/rejecting more items in each round will decrease fairness and efficiency**
  since less information on preferences is provided
- **Claiming/rejecting more items in each round will make procedure less vulnerable to manipulation**
  since less information is provide, less can be manipulated
- **Undercut will perform better than other splitting methods**

Outcome measures

Final allocation: $(S_{1,} S_2): S_1 \cap S_2 = \emptyset \,; S_1 \cup S_2 = S$

Utilities: $u_i(S_i) = \sum_{j \in S_i} v_{ij}$

Efficiency: relative sum of utilities:

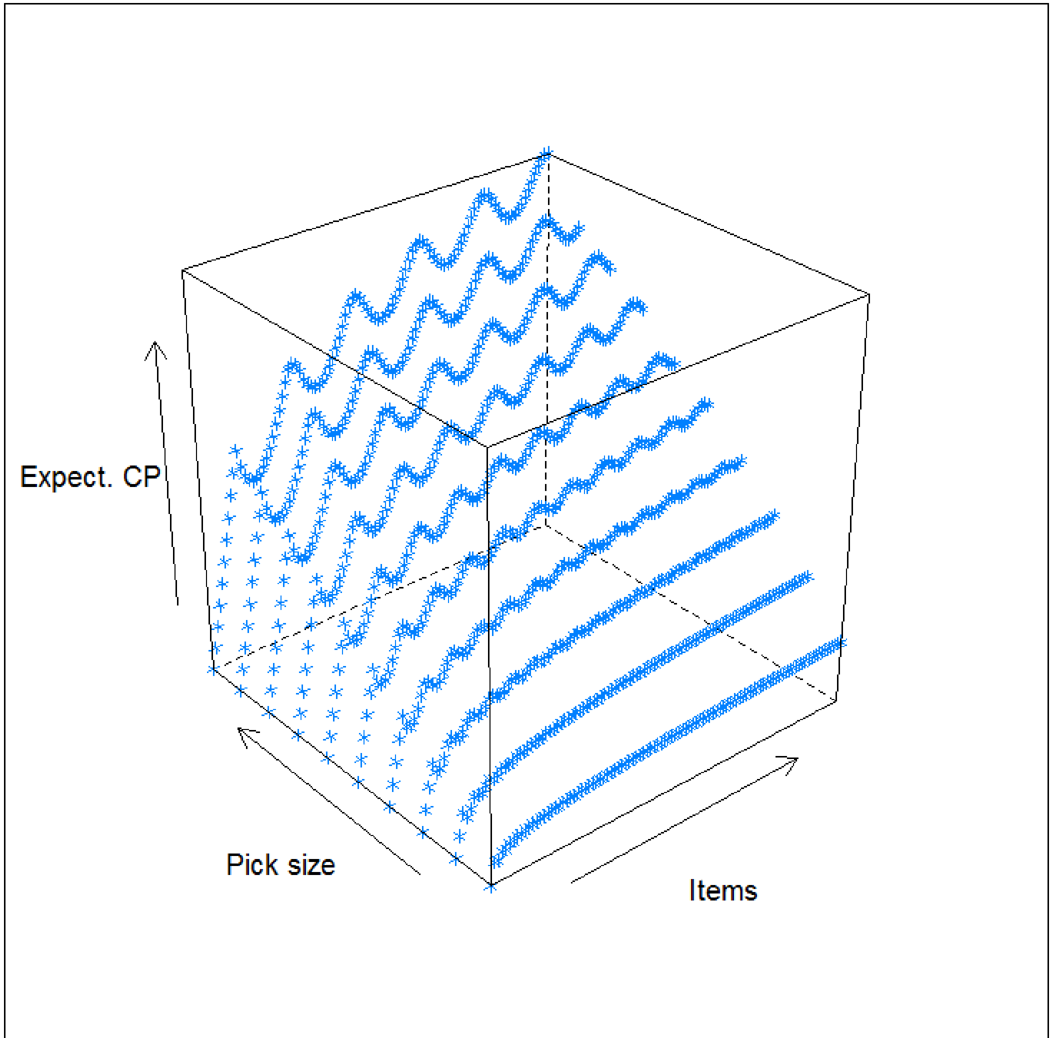
$$RelSum(S_{1,} S_2) = \frac{u_1(S_1) + u_2(S_2)}{\sum_j \max(v_{1j}, v_{2j})}$$

Fairness: relative position of weaker player

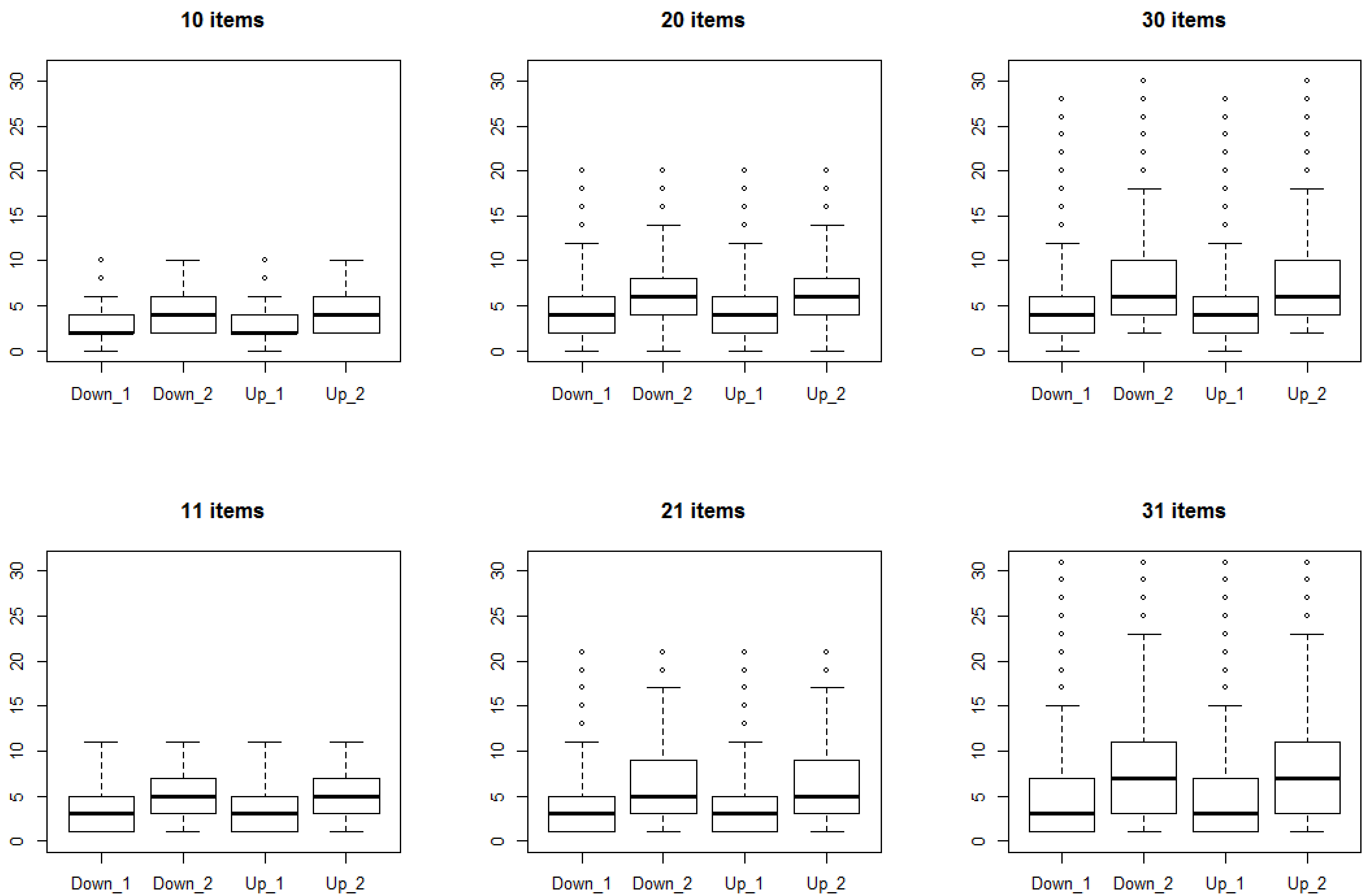
$$RelMin = \frac{\min(u_1(S_1), u_2(S_2))}{\min(u^*_{1,} u^*_2)}$$

$u^*_{1,}\ u^*_2$ … utilities in max-min allocation

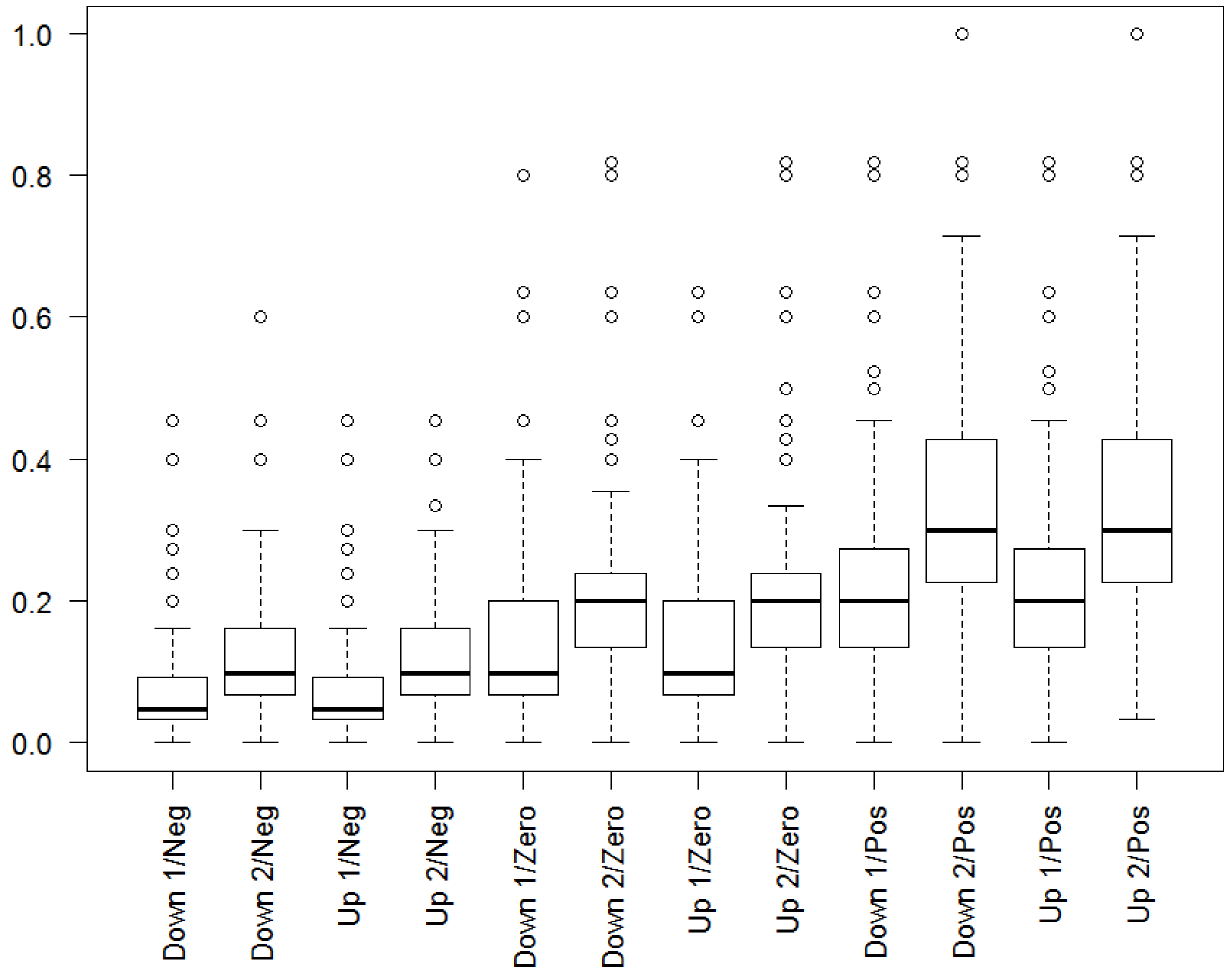
Expected size CP



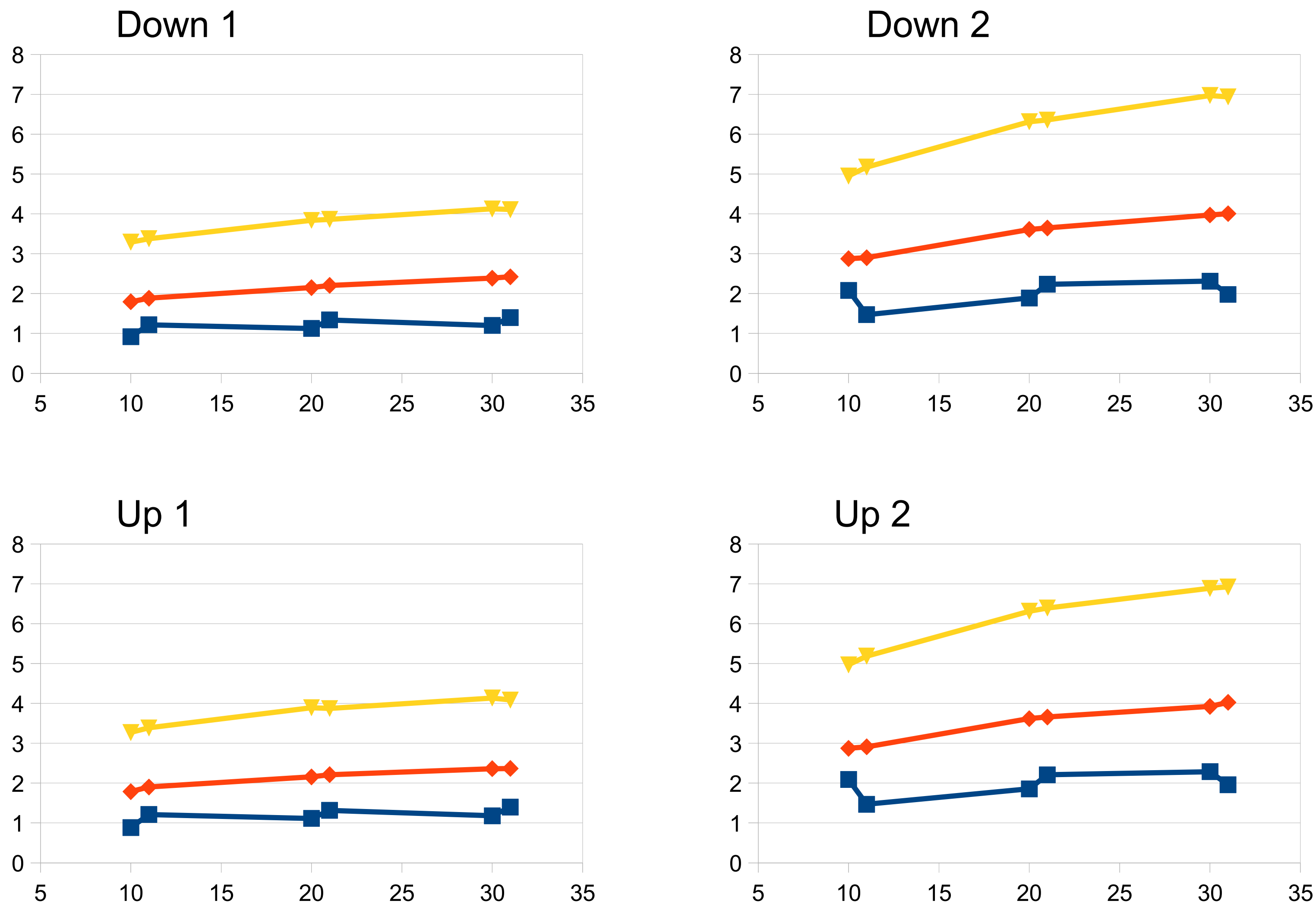
Size of contested pile



Relative size of contested pile



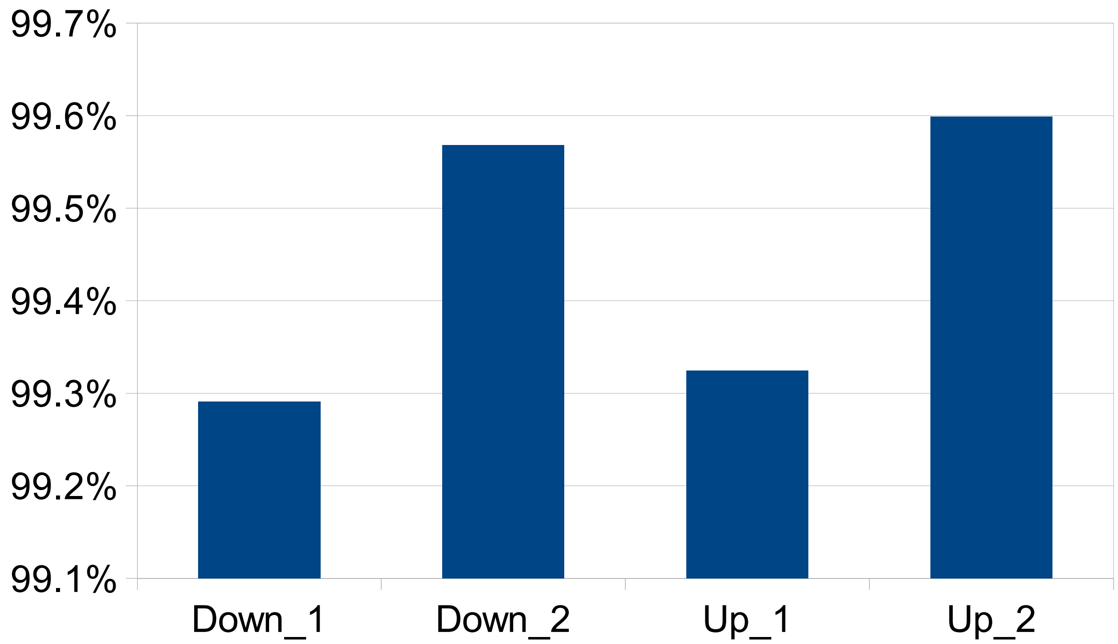
Absolute size of contested pile



Positive Zero Negative correlation

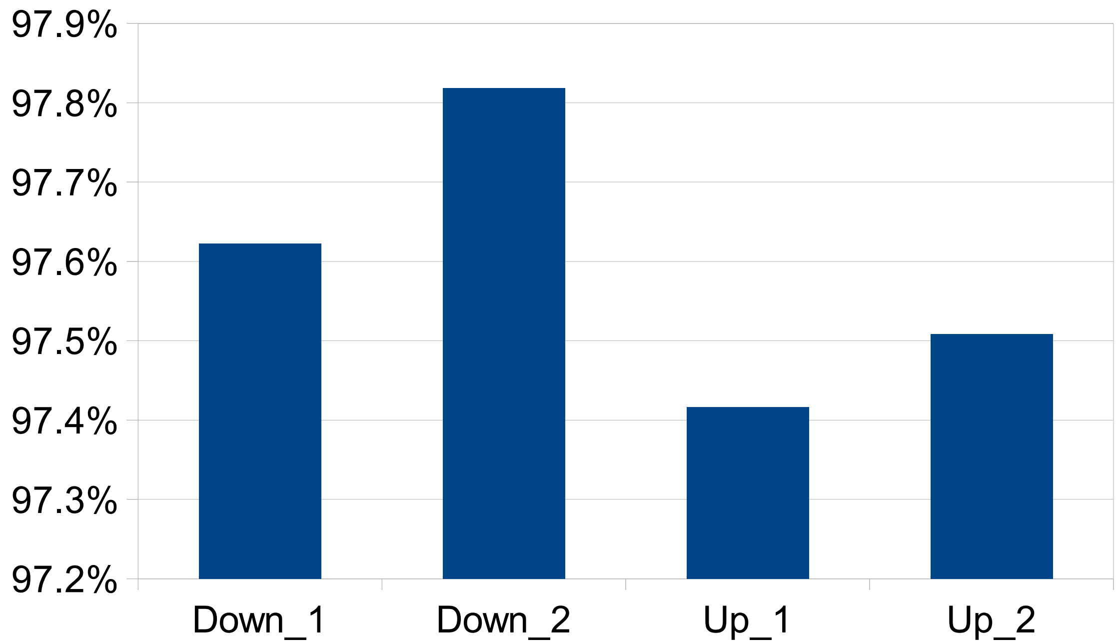
universität wien

Efficiency – Cases without CP



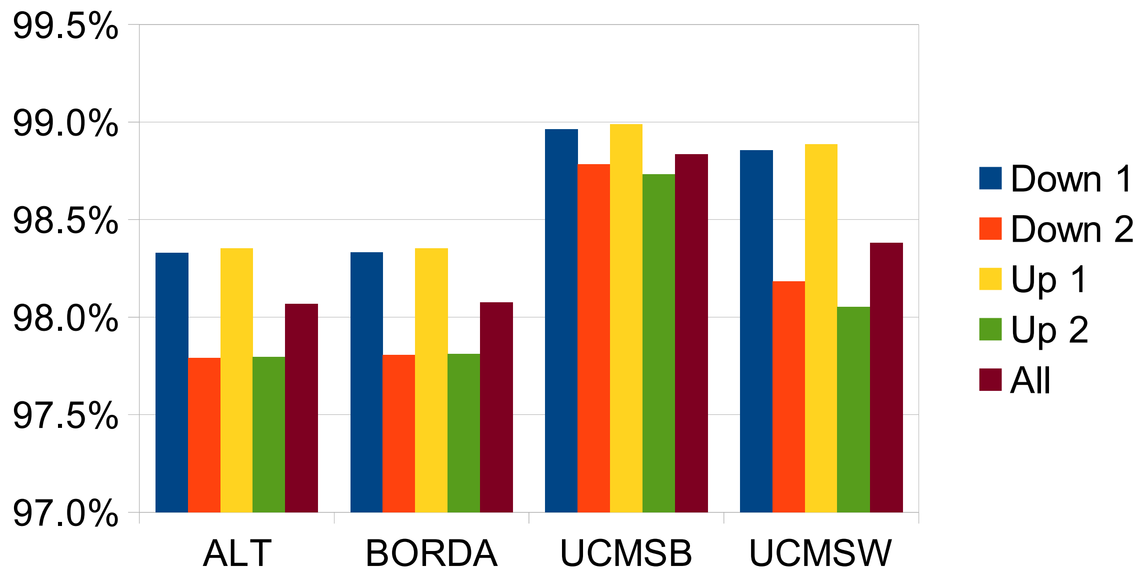
Larger pick size leads to higher efficiency?

Fairness - Cases without CP



- Larger pick size also better for fairness?
- Selecting is better than rejecting

Results Efficiency



Generation:
- Pick size 1 dominates 2
- Small differences between directions

Split:
- Undercut dominates other methods

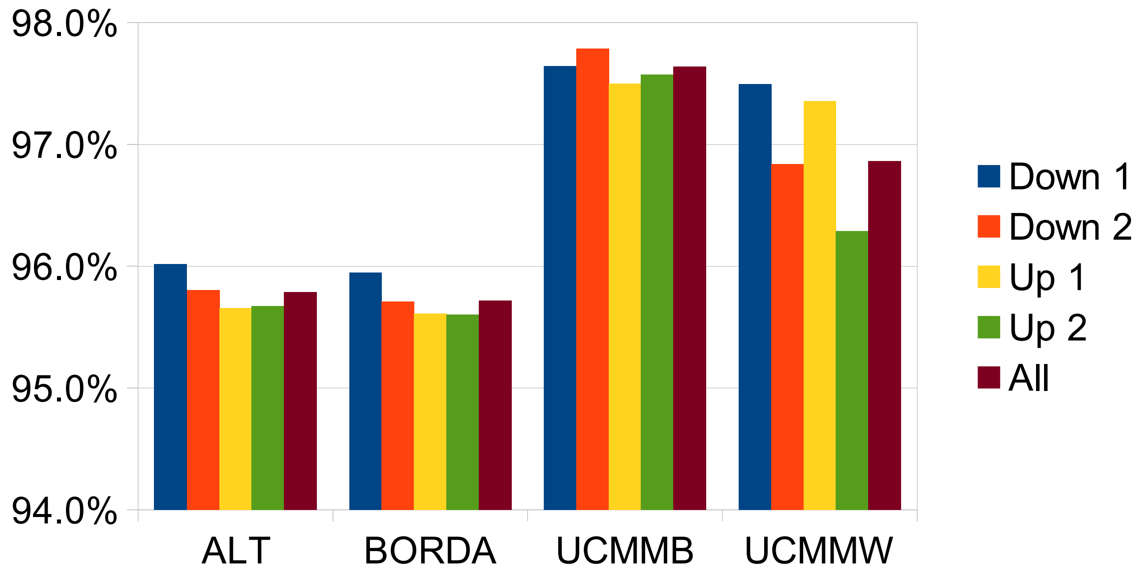
Discussion efficiency

No CP (no split): larger pick size better

With CP: Smaller pick size better

But: No CP will happen only if preferences are very distinct, and even more so with larger pick size:
→ No CP with larger pick size is “simple” problem

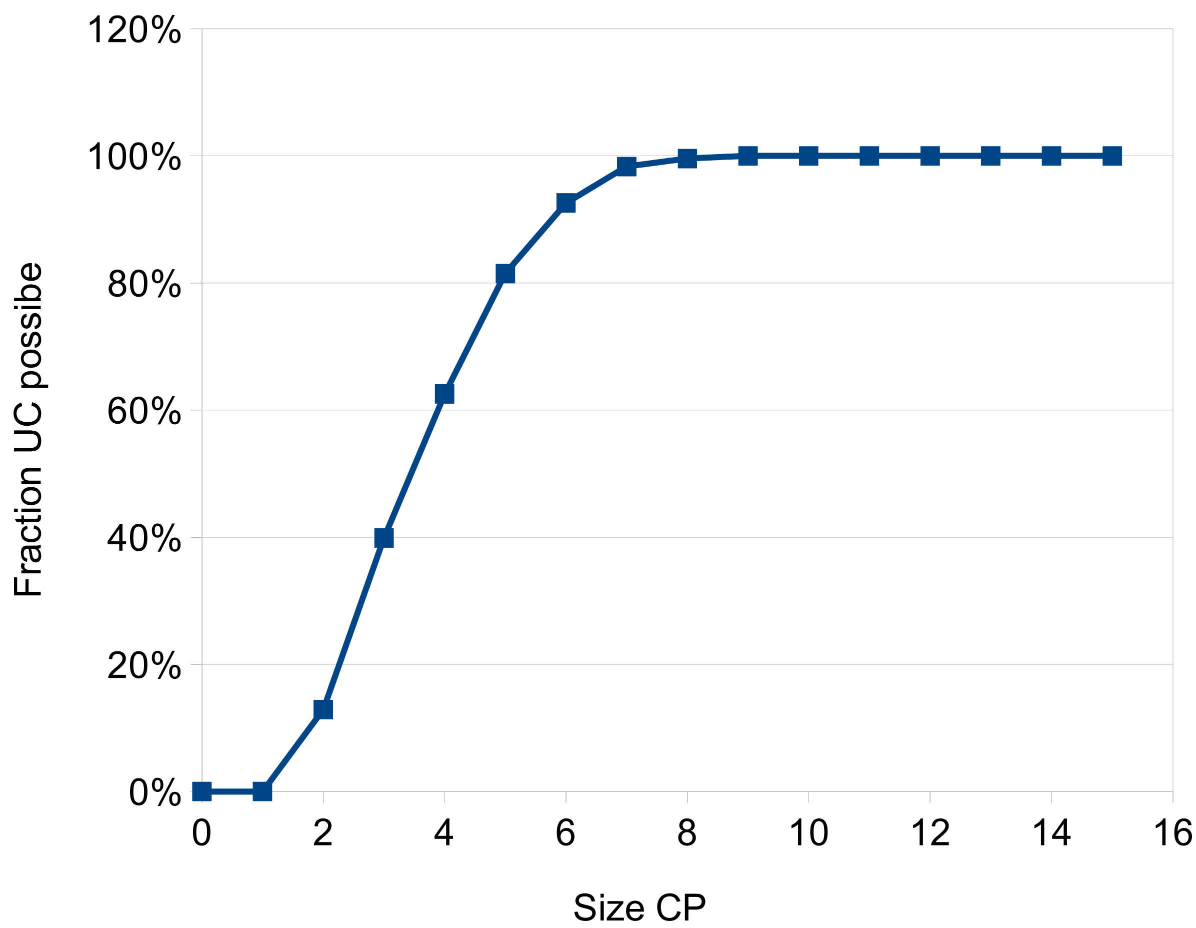
Results fairness



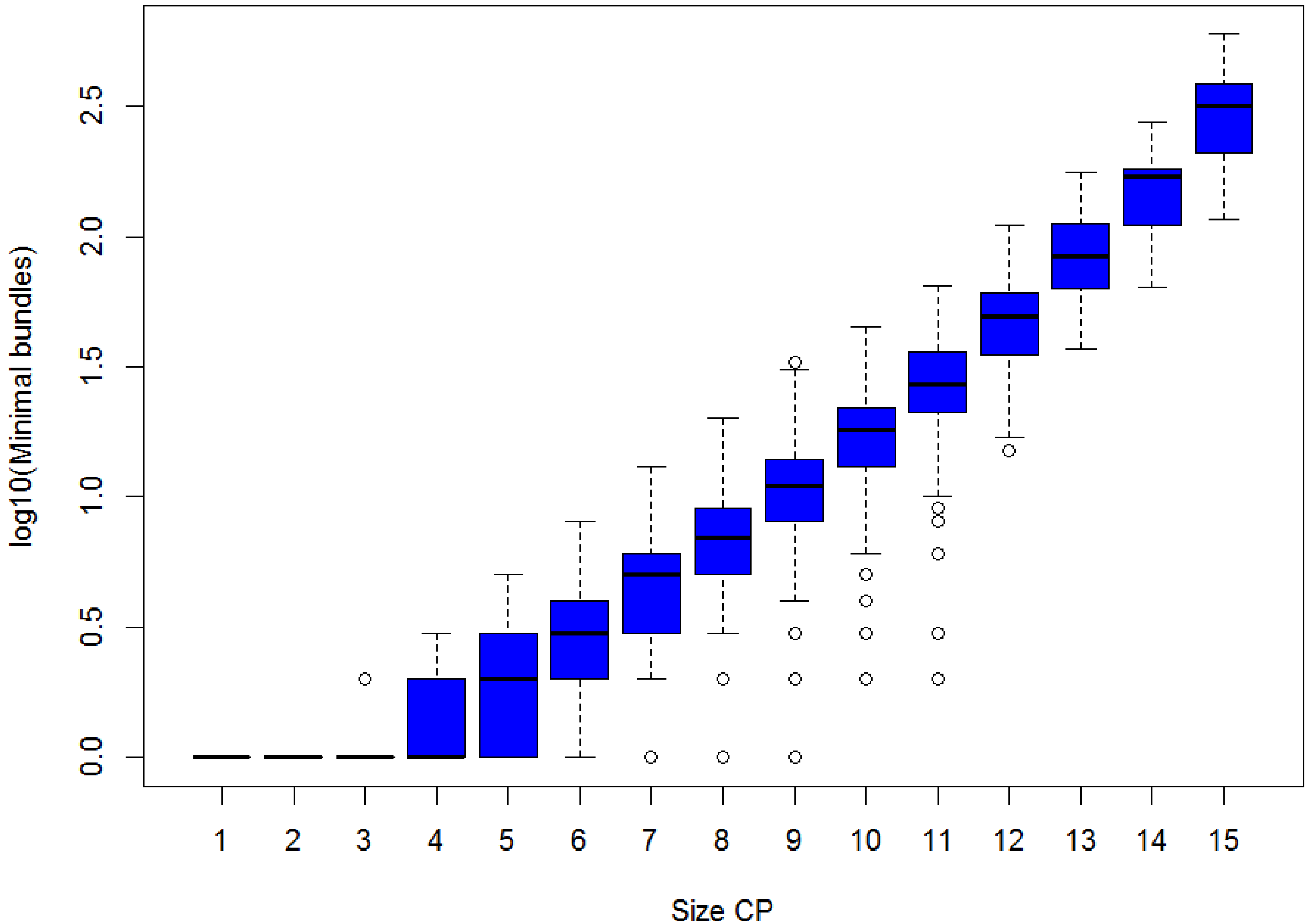
Generation:
- Downward dominates upward
- Effects of pick size mixed

Splitting
- Undercut dominates other methods

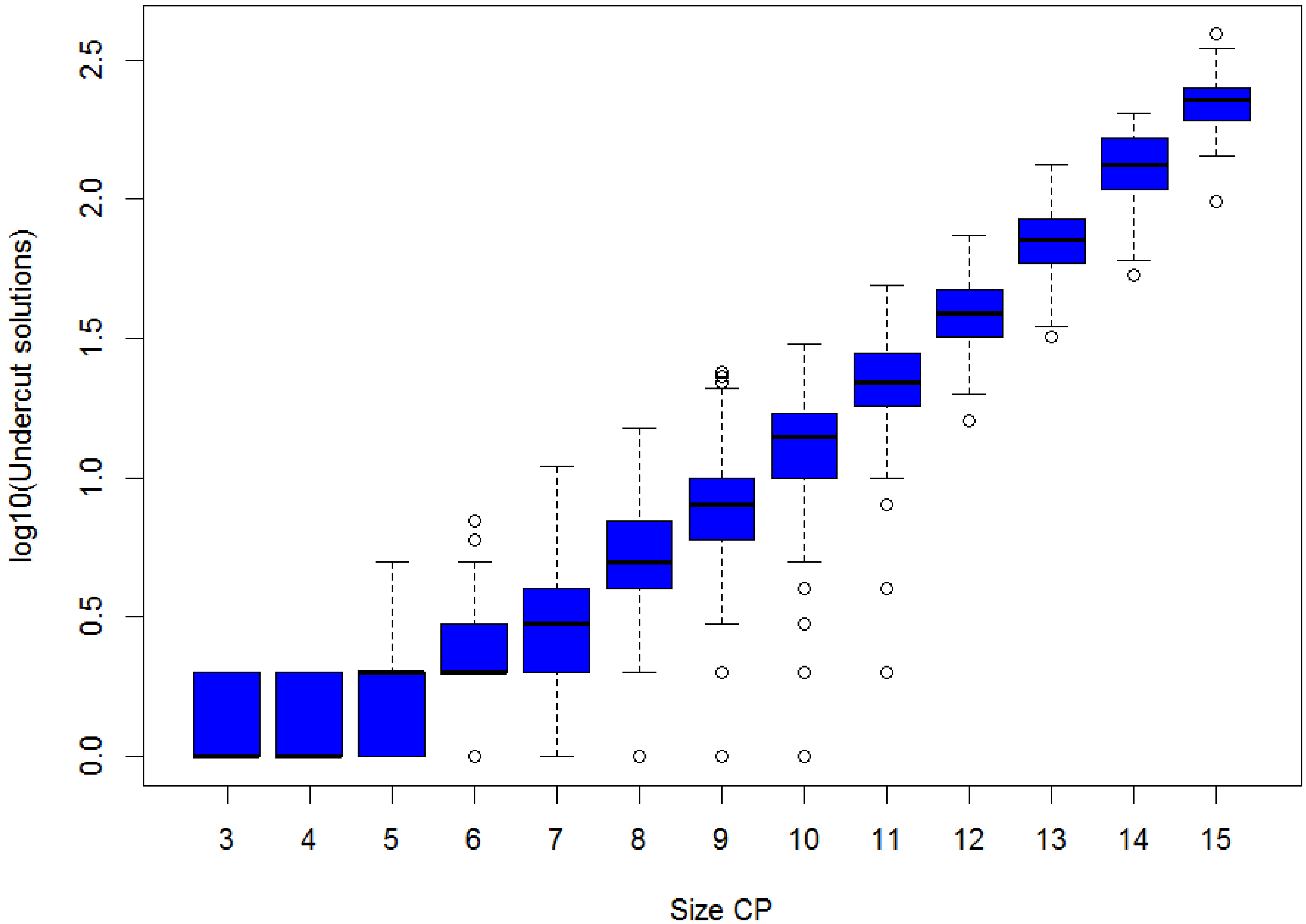
Undercut possible



Number minimal bundles



Number undercut solutions



Undercut evaluation

- Outperforms other splitting methods
- Requires certain size of CP (works 60% for nCP=4, 80% for nCP=5)
- … but will be demanding for larger CP sizes (≈ minimal bundles for nCP=13)
- Suitable for medium sized CP

Example

Item	u_{1i}	u_{2i}
A	4	1
B	3	3
C	2	4
D	1	2

Round	Player 1 claims	Player 2 claims	Result
1	A	C	Assigned
2	B	B	Contested
3	D	D	Contested

Note: both players prefer B over D

Strategic behavior

Item	u_{1i}	u_{2i}
A	4	1
B	3	3
C	2	4
D	1	2

Round	Player 1 claims	Player 2 claims	Result
1	B	C	Assigned
2	A	D	Assigned

Strategic: Player 1 gets A and B for sure
Sincere: Player 1 gets A for sure and either B or D

Player types

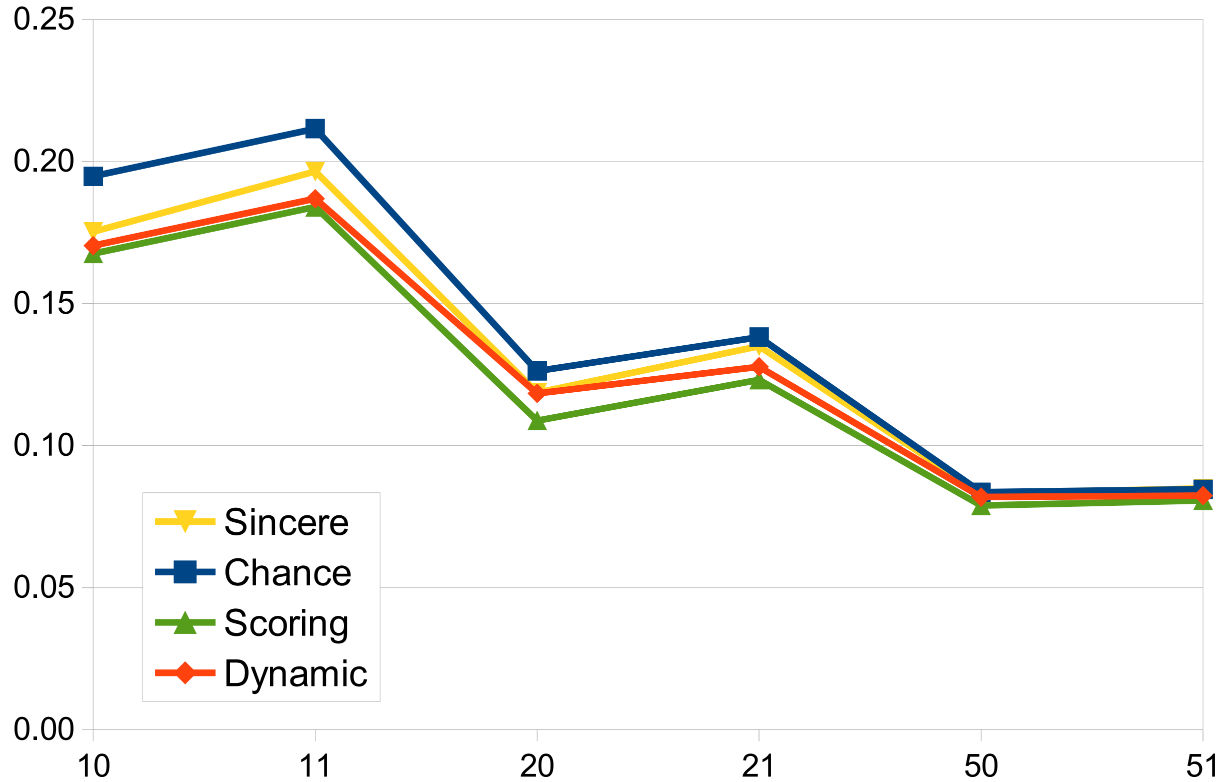
- **Sincere**
  always acts according to true preferences (i.e. selects n best items, rejects n worst items
- **Chance (m):**
  randomly select/reject out of the m best/worst items (with linearly decreasing probability according to true ranks)
- **Scoring (w):**
  select/reject items according to score**:**
- **Dynamic (w,d):**
  solve limited (w items, d rounds) dynamic programming problem to find (locally) optimal strategy
  (not used in experiments on design parameters)

Effects of strategic play

		Sincere	Chance	Scoring	Dynamic	All
Sincere		0.3243	0.3258	0.3175	0.3097	0.3193
Chance		0.3174	0.3188	0.3099	0.3040	0.3125
Scoring		0.3300	0.3303	0.3238	0.3212	0.3263
Dynamic		0.3355	0.3301	0.3216	0.3150	0.3256
Sincere vs. Chance	t	12.3268	12.9858	13.7982	10.0198	24.3488
	p	0.0000	0.0000	0.0000	0.0000	0.0000
Sincere vs. Scoring	t	-10.7658	-8.8210	-11.9415	-21.1370	-26.3143
	p	0.0000	0.0000	0.0000	0.0000	0.0000
Sincere vs. Dynamic	t	-21.6152	-8.5126	-7.8883	-9.4814	-23.3977
	p	0.0000	0.0000	0.0000	0.0000	0.0000
Chance vs. Scoring	t	-22.6950	-21.5393	-25.3351	-30.5677	-49.9217
	p	0.0000	0.0000	0.0000	0.0000	0.0000
Chance vs. Dynamic	t	-33.4020	-21.2479	-21.5547	-19.0602	-47.0422
	p	0.0000	0.0000	0.0000	0.0000	0.0000
Scoring. vs. Dynamic	t	-10.5638	0.3142	4.1852	11.2099	2.8589
	p	0.0000	0.7534	0.0000	0.0000	0.0043

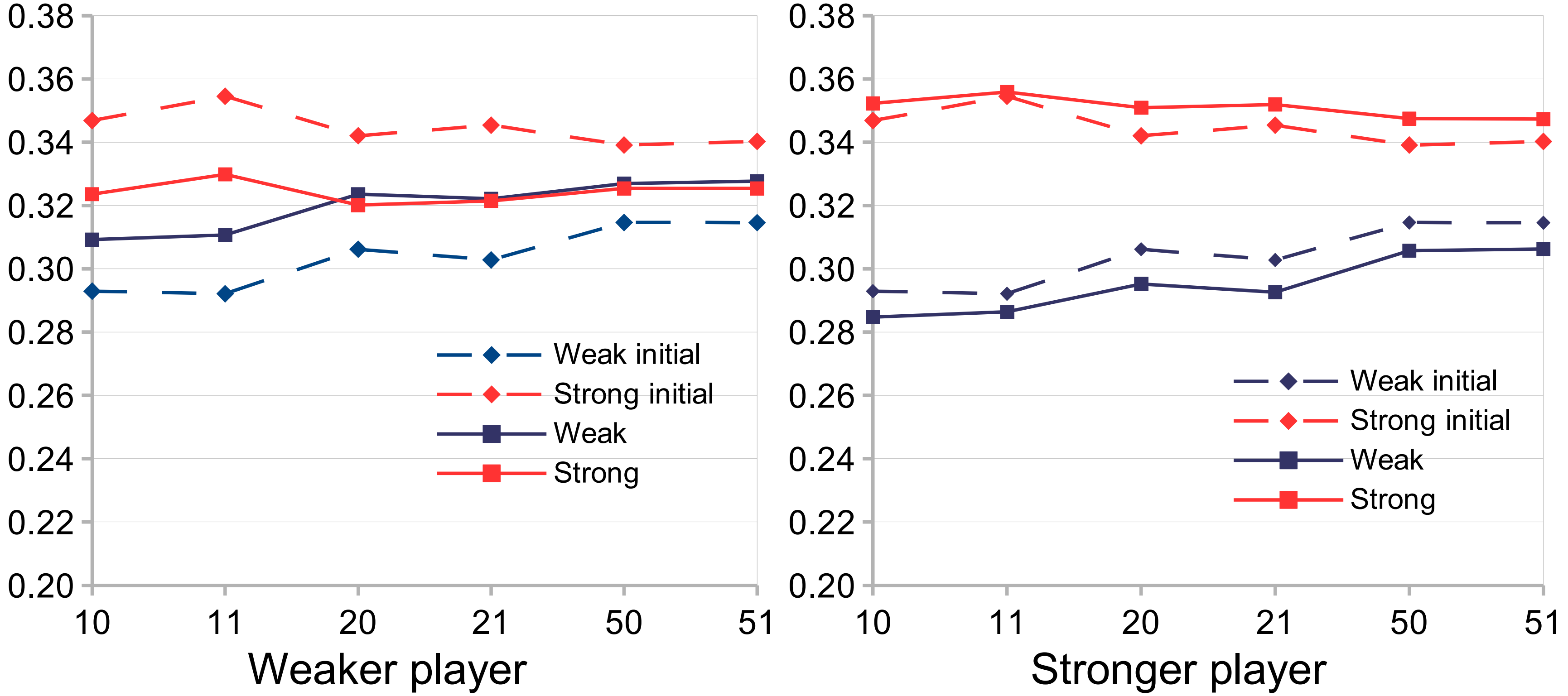
significant effect in opposite direction

Strategic play and fairness: Relative spread of utilities



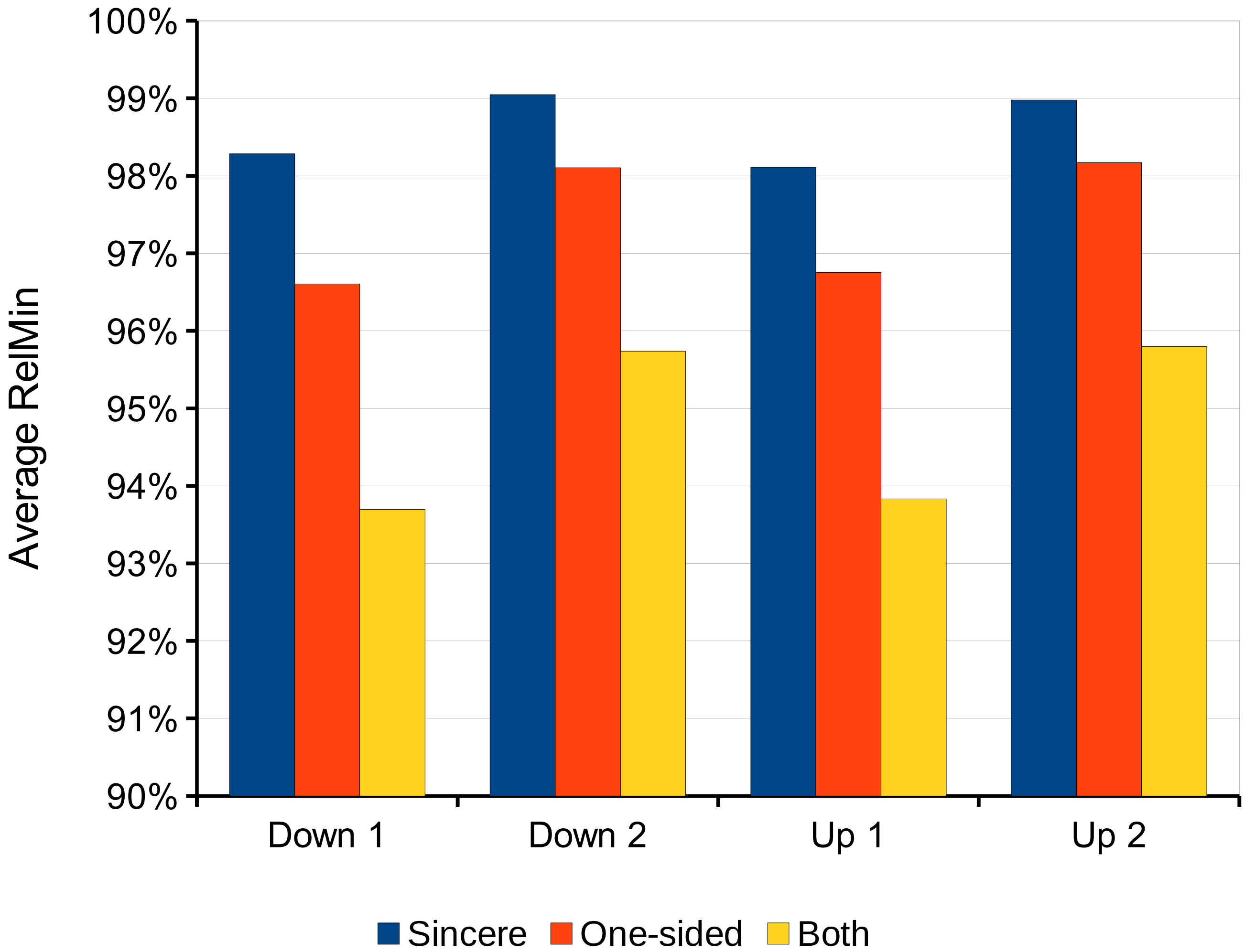
Items		10	11	20	21	50	51
Dynamic	t	-2.6027	-4.6570	-0.2227	-5.2406	-1.5182	-3.0812
vs. Sincere	p	0.0093	0.0000	0.8238	0.0000	0.1290	0.0021
Score vs.	t	4.1274	6.1591	8.2181	8.5957	5.1713	5.0399
Sincere	p	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Strategic play and fairness: Dynamic strategy and initial positions

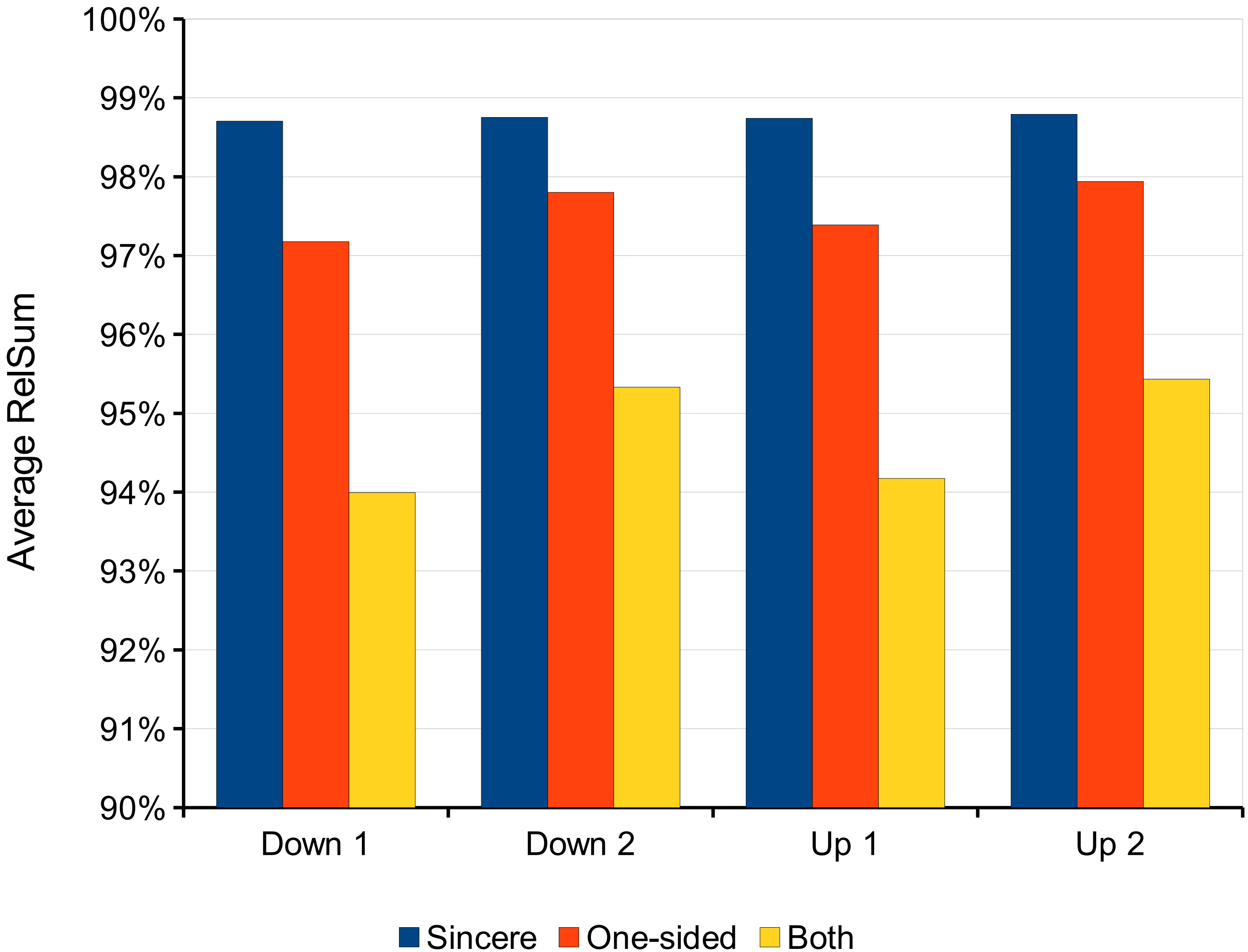


from sincere/sincere game switches to dynamic

Strategic play and design parameters: fairness



Strategic play and design parameters: efficiency



Conclusions

- **Information trade-off (small vs. larger pick size)**
  - More information improves both efficiency and fairness
  - But provides more room for strategic play
- **As expected, direction (chose/reject items) has no effect**
- **Undercut procedure for splitting CP**
  - Outperforms other methods
  - But has narrow field of application in terms of CP size

Extensions

- Incomplete information in strategic play
- Add cardinal preference information ("bidding")
- Other mechanisms for splitting the CP
- Efficient algorithms for undercut


Thank you for your attention!

Backup slides

Undercut procedure (1)
(Brams, Kilgour, Klamler 2012)

Set of items X

Preference relation P on items (same for both players if X is the CP)

Preferences on subsets of X are *responsive*:

$$x \in S, y \in X \setminus S : x R y \Leftrightarrow S \succ S \setminus \{x\} \cup \{y\}$$

(replacing an item with a less preferred item makes set less preferred)

$S \subseteq X$ is a *minimal bundle* iff

$$S \succ X \setminus S$$
$$\forall x \in S, y \notin S : X \setminus (S \setminus \{x\} \cup \{y\}) \succ S \setminus \{x\} \cup \{y\}$$

Undercut procedure (2)

1. Find set S, which is a minimal bundle for one player (w.o.l.g player 1), but not for the other player
2. Propose to allocate S to player 1 and $X \setminus S$ to player 2
3. Player 2 can "undercut"
   let for both players $xPy, x \in S, y \in X \setminus S$
   Player 2 takes $T = S \setminus \{x\} \cup \{y\}$
   Player 1 then receives $X \setminus T$

Undercut procedure (3)

Resulting allocation is envy-free:
S is minimal bundle for player 1, but not for 2

Case 1:
Player 2 prefers complement over S:
both get what they prefer

Case 2:
S is more than minimal bundle for player 2:
Player 2 can reduce S to a bundle T, which he still prefers over its complement
Since S was minimal for player 1, player one prefers $X \setminus T$ over T

Expected size of CP

Notation
m.... items in total
h..... items left to allocate in a given round
q..... pick size
r..... items contested in one round

Regular (not last) round:
$q - r$ items to player 1
$q - r$ items to player 2
r items contested
$2q - r$ items allocated in total

Process terminates once $h = 2q - r$
(r must always be large enough for equality)

Expected CP size

Probability that r items will be put on CP:
w.l.o.g. player 1 picks items $1..q$

$h>2q:$ zero to q items can be contested

$$Pr_h(r)=\binom{r}{q}\binom{q-r}{h-q}/\binom{q}{h}$$

$q<h<2q:$ at least $2q-h$ items will be contested

$$Pr_h(r)=\begin{cases} 0 & r<2q-h \\ \binom{r}{q}\binom{q-r}{h-q}/\binom{q}{h} & 2q-h\leq r\leq q \end{cases}$$

$h\leq q:$ all items are contested

$$Pr_h(r)=\begin{cases} 0 & r\neq h \\ 1 & r=h \end{cases}$$

Expected CP size

Probability that CP contains i items

$$Pr_m(i) = \sum_{r=\max(2q-m,0)}^{q} Pr_{m-2q+r}(i-r) \binom{r}{q} \binom{q-r}{m-q} / \binom{q}{m}$$

Expected size of CP: Example m=6, q=2