On the Division of Indivisible Items

Steven J. Brams, NYU

D. Marc Kilgour, Wilfrid Laurier Univ.

Christian Klamler, University of Graz

COST Action IC 1205 – Meeting
Sibiu, 20 October 2014

Introduction – General Aspects

What are typical fair division problems?

land division

cake cutting

cost/surplus sharing

dividing sets of items

Introduction

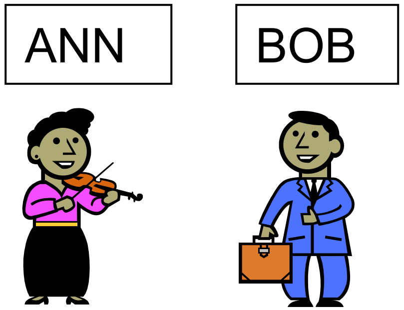
Most of the fair division problems have a similar formal structure.

- What is to be divided?
  - costs, cakes, indivisible goods, etc.
  - possible restriction, e.g. in form of network structures, etc.
- What do agents' preferences look like?
  - depends on the information acceptable in the division process
  - claims, rankings of items, cardinal value functions, etc.
- How are we dividing? What do we want to achieve?
  - define rules of a fair division procedure
  - what properties do such procedures satisfy
    - used to define fairness

- many surveys
  - Moulin (2003)
  - Thomson (2008)
  - Brams (2006)
  - Bouveret, Chevaleyre and Maudet (2015)

Indivisible Goods

Here we consider the problem of **fairly dividing** a set of **indivisible items** between **two (or more) players**.



Examples:

- divorce settlement
- inheritance problems
- allocations of tasks to workers

- Assumptions
  - only ordinal preference information over set of items
    - weaker (but probably more realistic) than attaching utilities to items
  - no synergies among the items (neither positive nor negative)
  - no monetary transfers

Related Literature and Outlook

Related Literature:

- Brams and Taylor (1996); Brams, Edelman and Fishburn (2003)
  - general procedures
- Barberà, Bossert and Pattanaik (2004)
  - ranking sets of items
- Bouveret and Lang (2011); Bouveret, Endriss and Lang (2010), Aziz et al. (2013); Lipton et al. (2004); Procaccia and Wang (2014); Bouveret and Lemaitre (2014)
  - procedures and computational aspects

- In this presentation
  - fair division procedures
  - contested pile procedure
  - Brams, Kilgour and Klamler (2012, 2014)

Formal Framework

- Assume set X of m items ranked by the players
- $P_i \subset X \times X$ as player i's strict preference over X
  - no further information used
- $\mathcal{X}$ denotes the set of all subsets of X
- $\succsim_i$ as i's preference over $\mathcal{X}$
  - no synergies between preferences!
- Example: X = {a,b,c,d}; N = {1,2}

P_1	P_2
a	b
b	c
c	d
d	a

Brams & Taylor Procedure (1999)

- Consider the following simple procedure (BT-procedure):
  - ask players to name the item they want to have next
  - if they name different items allocate them
  - if they name the same item put it into contested pile

- might lead only to partial allocation

- but does it satisfy desirable properties?
  - envy-freeness
  - efficiency
  - completeness

P_1	P_2
a	b
b	~~c~~
~~c~~	~~d~~
~~d~~	a

Allocation:
S_1 = {a}; S_2 = {b}; CP = {c,d}

Envy-freeness

P_1	P_2
a	b
b	c
c	d
d	a

- usual definition of envy-freeness
  - an allocation (S_1,S_2) is EF if for all $i \in N$, $S_i \succsim_i S_j$
  - this is the case in the previous example as $\{a\} \succ_1 \{b\}$ and $\{b\} \succ_2 \{a\}$

- our definition
  - as we use no information other than the players' rankings

An allocation (S_1,S_2) is EF iff there exist an injection f_1: $S_1 \to S_2$ and an injection f_2: $S_2 \to S_1$ such that for each $x \in S_1$, $x \succ_1 f_1(x)$ and for each $x \in S_2$, $x \succ_2 f_2(x)$.

  - hence we have EF if there is pairwise dominance (Bouveret, Endriss and Lang (2010))
  - possible and necessary envy-freeness
  - for P1 in above example: $\{a,c\}$ is necessarily EF whereas $\{a,d\}$ is possibly EF

Completeness

- when can we be sure that a complete EF allocation (S_1, S_2) does exist, i.e., all items can be allocated in an envy-free way?

> Condition C(k): A set consisting of i's k-most preferred items is equal to the set consisting of j's most preferred items.

  - only concerned with equality of sets not with their rankings
  - it will be important whether this condition holds for odd k

> Condition D: Condition C(k) fails for all odd values of k ≤ m.

P_1	P_2
a	b
b	d
c	f
d	a
e	c
f	e

k=1: {a} vs {b}
k=3: {a,b,c} vs {b,d,f}
k=5: {a,b,c,d,e} vs {b,d,f,a,c}

Result

Theorem: Let m be even. A pair of strict preference rankings of m items admits a complete EF allocation iff it satisfies Condition D.

- alternative conditions for complete EF possible
  - for any $k \leq m$, the number of items assigned to the other player up to k is at most $k/2$

Alternative Procedure

- Consider the following procedure (AL-procedure):
  - players communicate their strict preferences over X
  - stage 0
    - compare their most preferred unallocated items
    - if identical put into contested pile → repeat stage 0
    - if different allocate → go to stage t = 1
  - stage t
    - if no item remains → stop; if one item remains put it in CP → stop
    - if most preferred unallocated items are different → assign and go to stage t+1
    - if most preferred unallocated item is the same → perform feasibility check
    - if feasibility check negative → put item in contested pile → repeat stage t
    - if feasibility check positive → assign items accordingly → go to stage t+1

- feasibility check
  - assign item i to P1 and next best item (compensation item) in P2's ranking to P2
  - check whether the number of items assigned so far to P1 (including the current item) which are considered better by P2 than P2's comp. item is at most t
  - make same check for roles interchanged (multiple outcomes possible!!)
  - check is positive if one of the cases feasible, otherwise negative

AL Procedure

- Example

P_1	P_2
a	b
b	c
c	d
d	a

- stage 0
  - a to P1; b to P2
- stage 1
  - both request item c
  - feasibility check for c to P2 and d to P1 negative
  - feasibility check for c to P1 and d to P2 positive
- *AL-assignment: S_1 = {a,c}; S_2 = {b,d}*
- *AL gives complete EF-allocation whereas BT only partial EF-allocation*

AL Procedure

- Example

P_1	P_2
a	b
b	c
c	e
d	d
e	a
f	f

- BT-allocation
  - S_1 = {a,d}; S_2 = {b,e}; CP = {c,f}
- AL-allocation
  - S_1 = {a,c}; S_2 = {b,e}; CP = {d,f}

AL-Procedure - Results

The number of items allocated under AL is never less, and may be more than under BT. If the number is the same, but some items differ, then the AL allocation Pareto dominates the BT allocation.

An AL allocation is a maximal EF allocation: There is no other EF allocation that allocates more items to the players.

- Local Pareto optimality (LPO)
  - an allocation is LPO if there is no other allocation of the same items between the players that Pareto dominates it

Both, BT and AL, produce LPO allocations.

- but (not surprisingly)

Both, BT and AL, are manipulable.

Extensions - other procedures

- sequential procedures (Bouveret and Lang, 2011)
  - use sequences, e.g. 121212..., or 12211221..., etc.
  - what is the fairest sequence?
    - full independence of preferences
    - utilitarianism – egalitarianism under Borda scores

egalitarian	utilitarian
1221	1212
121221	121212
12212112	12121212
1221121221	1212121212
121212122121	121212121212

Extensions

- maximin shares (Procaccia and Wang, 2014)
  - cut and choose leads to problems
  - no guarantee of 1/n – share
  - maximin share as what a player can guarantee herself by dividing the items in n piles

Theorem: There exists an allocation $(S_1,S_2,\ldots,S_n)$ such that $u_i(S_i) \geq 2/3 \text{ MMS}_i$

- can be found in polynomial time
- www.spliddit.org

Extensions

- descending demand procedure (Herreiner and Puppe, 2002)
  - players rank all their bundles
  - descend in their rankings until PO and maximin-optimal allocation is found
  - does not guarantee EF but produces "balanced" allocations
  - ranking of all bundles realistic?

- adjusted winner procedure (Brams and Taylor, 1996)
  - assign (100) points to items
  - transfer items to equalize sum of points
  - one item may have to be divided

Theorem: The adjusted winner procedure leads to an allocation which is EF, PO and equitable.

Contested Pile

- BT and AL might lead to non-empty contested pile
- what could we do with the items in the contested pile?
- before going in detail, consider the ultimatum game of dividing a single divisible good

ANN

BOB

- now – in a second stage – allow Bob to undercut Ann's proposal by 1 cent and implement the resulting division
  - what will Ann do in the first stage under these conditions?

Contested Pile

- are we able to divide the items in the contested pile even if both players rankings of the items are the same?
  - as is the case in BT and AL

Is there a fair division procedure that leads to an envy-free division?
(at least under certain restrictions)

Definitions

- Preference $\succeq$ on $\mathcal{X}$ satisfies responsiveness if for all $S \in \mathcal{X}$ and all $x \in S$ and $y \in X \setminus S$

$$xRy \Leftrightarrow S \succ S \setminus \{x\} \cup \{y\} \quad \text{and} \quad S \succ S \setminus \{x\}$$

P_1
a
b
c
d
e

$\{a,b,e\} \succ \{b,c,e\}$

- Let $S, T \in \mathcal{X}$. T is said to be ordinally less than S, denoted by $T \leq_{OL} S$, if there exists an injective function $\sigma_{T,S}: T \setminus S \rightarrow S \setminus T$ such that for all $x \in T \setminus S$, $\sigma_{T,S}(x) P x$.

Definitions

- $S \in \mathcal{X}$ is a minimal bundle for player i if $S \succeq_i -S$ and, for any $T \leq_{OL} S$, it holds that $-T \succ_i T$
- Player i regards set $S \in \mathcal{X}$ as worth at least 50 percent if $S \succeq_i -S$

➔ Hence, a player regards a subset S as a minimal bundle if S is worth at least 50 percent AND any subset T that is ordinally less than S is worth less than 50 percent.

$$\frac{P}{\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array}}$$

if {a,c,e} is a minimal bundle, then {a,d,e} must be worth less than 50%

Definitions

- For any $S \in \mathcal{X}$, the split $(S,-S)$ is envy-free if $S \succeq_1 -S$ and $-S \succeq_2 S$
- An envy-free split of X, $(S,-S)$, is trivial if $S \sim_1 -S$ and $-S \sim_2 S$

Undercut Procedure

- Given the CP, players state their sets of minimal bundles (MB_i)
- $MB_1 \neq MB_2$: randomly choose a player (say player 1) and let her propose a minimal bundle $S \in MB_1$ such that $S \notin MB_2$
- $MB_1 = MB_2$: if there exists an S such that $S, -S \in MB_i$ then S becomes the proposal; if no such minimal bundle exists, choose one at random as the proposal
- Given the proposal, the other player (say player 2) can either
  - accept the complement of the proposal
  - reject and undercut, i.e., take a set which is ordinally less than the proposal in which case its complement is assigned to the other player

Undercut Procedure

P
a
b
c
d
e

- assume {a,b} $\in MB_1$ and {b,c,d,e} $\in MB_2$ but not vice versa
- assume P1 makes the proposal

- P1 proposes {a,b}
- P2 can do the following
  - accept: she gets {c,d,e}
  - undercut: she takes {a,c} and P1 gets {b,d,e}
- P2: {c,d,e} must be worth less than 50% as {b,c,d,e} $\in MB_2$
- P2: {a,c} worth at least 50% as {b,d,e} ordinally less than {b,c,d,e}
- P1: {b,d,e} worth at least 50% as {a,b} $\in MB_1$ and therefore {a,c} less than 50% which makes the complement {b,d,e} worth more than 50%
- allocation ({b,d,e},{a,c}) is envy-free

Undercut Procedure - Result

Theorem

There is a nontrivial envy-free split of the contested pile if and only if one player has a minimal bundle that is not a minimal bundle of the other player. If so, UP implements an envy-free split.

- Different sets of MBs necessary, otherwise more information about players' preferences required.
- Definition (extension monotonicity):

Preference $\succsim$ on $\mathcal{X}$ satisfies extension monotonicity if for all $S, T \in \mathcal{X}$, all preferences P, and all $x, y \in X \backslash (S \cup T)$, $S \succsim T$ and xPy imply $S \cup \{x\} \succ T \cup \{y\}$.

Proposition

Given responsive and extension monotonic preferences of the players and an envy-free division of the contested pile, the final division of X under UP and any previous procedure is envy-free.

Undercut Procedure - Properties

- various properties have been analyzed
  - feasibility of EF allocations
    - condition to determine feasible subsets
  - size of contested pile
    - grows only moderately (under certain assumptions)
  - efficiency
    - might not be efficient (if certain underlying cardinal preferences are assumed)
  - manipulability
    - strategy-proof (based on maximin behavior)
    - but non-sincere behavior may be a Nash-equilibrium

- ➔ **more details in upcoming talk by**

Conclusion

Based on ordinal preferences we tried to investigate:

- What possibilities are there to find envy-free divisions of indivisible items
  - condition D
- AL procedure as a practical option based on purely ordinal preferences
  - envy-free, PO and maximal allocation
  - non-empty contested pile
- Undercut procedure as practical procedure to divide a contested pile
  - if the players' minimal bundles are not the same, the allocation will be envy-free
  - satisfies various interesting properties