Approximability of optimal social welfare in multiagent resource allocation with cardinal and ordinal preferences

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This is in part joint work with Dorothea Baumeister , Jörg Rothe, Sylvain Bouveret, Jérôme Lang, Trung Thanh Nguyen, and Abdallah Saffidine.

Meeting of COST Action IC 1205

Sibiu, October 20–22, 2014

Outline

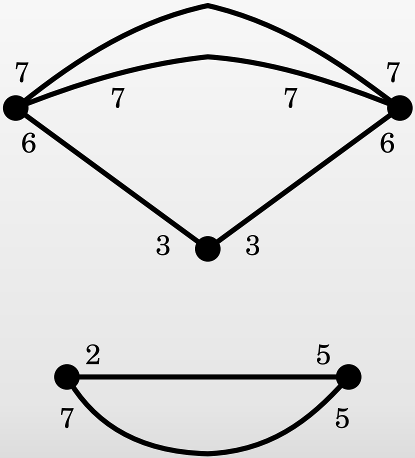
Resource allocation under...

1. Cardinal preferences
2. Ordinal preferences & Restricted model

Part One

Cardinal preferences

Example



Undirected graph with

- parallel edges
- self loops
- nonuniform weights

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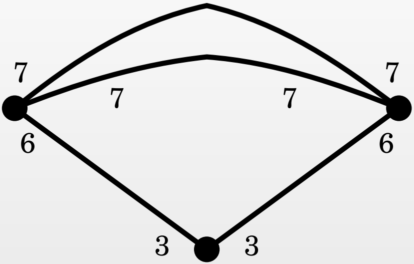

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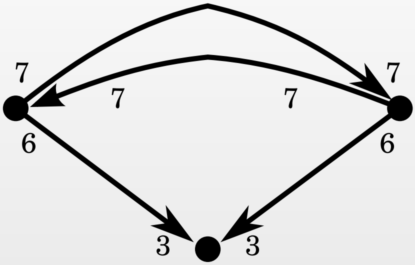

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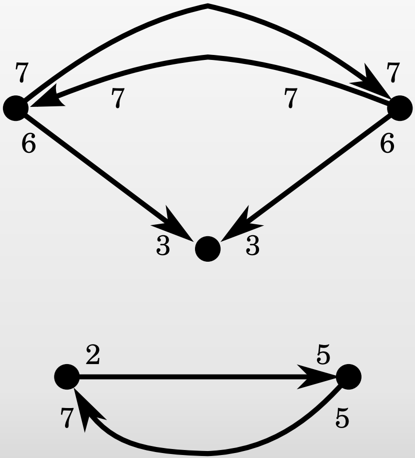

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- vertex = agent
- edge = object
- sum of incoming weights = utility

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What kind of resource allocation problem do we deal with?

- indivisible and non-shareable goods
- centralized
- no payments
- non-strategic agents

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Formal model:

- a finite set of **objects** $\mathcal{O} = \{o_1, \ldots, o_m\}$
- a finite set of **agents** $\mathcal{A} = \{1, \ldots, n\}$
- each agent $i \in \mathcal{A}$ has utility function $u_i : 2^{\mathcal{O}} \to \mathbb{Q}$

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Problem: Utility functions have an exponential-size domain.

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- in the ***k*-additive form** by
  coefficients α_i^S for each bundle $S \subseteq \mathcal{O}$ with $\|S\| \leq k$ such that

$$u_i(T) = \sum_{S \subseteq T, \|S\| \leq k} \alpha_i^S.$$

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Other representations are e.g., **straight-line programs**, **bidding languages**, **weighted goals**.

Restrictions on utility functions

Usually we have assumptions such as...

- no **externalities**
- **monotonicity** (free disposal)

$$S \subseteq T \implies u(S) \leq u(T),$$

- **normalization**

$$u(\emptyset) = 0.$$

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We can ask questions such as

- Which **(fairness) properties** are satisfied?
- What is the **social welfare**?

Social welfare

We can aggregate utility values with a **collective utility function**.

- **Utilitarian social welfare**

$$sw_u(\pi) = \sum_{i \in \mathcal{A}} u_i(\pi)$$

- **Egalitarian (Rawlsian) social welfare**

$$sw_e(\pi) = \min_{i \in \mathcal{A}} u_i(\pi)$$

- **Nash product**

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There are also approaches using inequality indices, e.g., Gini index.

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	o_1	o_2	o_3
agent 1	5	1	5
agent 2	1	4	6

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Egalitarian social welfare:

$$\pi = \langle \{o_1\}, \{o_2, o_3\} \rangle \rightarrow sw_e(\pi) = \min(5, 4 + 6) = 5$$

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$\pi = \langle\{o_1\}, \{o_2, o_3\}\rangle \rightarrow sw_N(\pi) = 5 \times (4 + 6) = 50$

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$$\pi' = \langle\{o_1, o_2\}, \{o_3\}\rangle \to sw_N(\pi') = (5+1) \times 6 = 36$$

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k-additive form, $k \geq 1$:

Utilitarian (except for $k = 1$), egalitarian, and Nash product social welfare optimization are **hard**.

But for $m = n$ it is easy.

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- Approximate notions of properties
- Suboptimal social welfare
- Restricted model of computation (Part two)

Utilitarian social welfare

Bundle form	Approximability	Reference
general	NP-hard in factor $m^{\varepsilon-1/2}$	[LOS99]
submodular	$1-(1/e)$ NP-hard in factor $1-(1/e)+\varepsilon$	[FGMS06],[CCPV07],[Von08] [KLMM08]
subadditive	hard in factor $1/m^{1/4}$ $1/m^{1/2}$	[DS06] [DNS10]

Egalitarian social welfare

Bundle form	Approximability	Reference
general	NP-hard in any factor	[NRR13]
submodular	$1/(m-n+1)$	[Gol05]
	$1/(m^{1/2}n^{1/4}\log m \log^{3/2} n)$	[GHIM09]
subadditive	$1/(2n-1)$	[KP07]

k-additive form	Approximability	Reference
1-additive	NP-hard in factor $1/2+\varepsilon$ $1/m^{\varepsilon}, \epsilon \in \mathcal{O}(1)$	[BD05] [CCK09]
1-additive, Santa Claus	$\mathcal{O}(1)$	[Fei08],[HSS11]
k-additive, $k \geq 2$	NP-hard in any factor	[NRR13]

Nash product

Restriction	Approximability	Reference
Bundle form	NP-hard in any factor	[NRR13]
1-additive	$1/(m-n+1)^n$	[NR13]
2-additive	NP-hard in factor $21/22+\varepsilon$	[NRR13]
k-additive, $k \geq 3$	NP-hard in any factor	[NRR13]

Part Two

Ordinal preferences & Restricted model

Our approach

Problem: What to do if no clear numerical scale (*e.g* money) is involved...?

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Idea: *Cardinalize ordinal preferences with the help of scoring vectors.*

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k-Approval	1	1	0	0	0	0

Example

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5 objects, 3 agents...

- 1 : $o_1 \succ o_2 \succ o_3 \succ o_4 \succ o_5$
- 2 : $o_4 \succ o_2 \succ o_5 \succ o_1 \succ o_3$
- 3 : $o_1 \succ o_3 \succ o_5 \succ o_4 \succ o_2$

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Let's consider allocation $\pi = \langle \{o_1\}, \{o_4, o_2\}, \{o_3, o_5\} \rangle$.

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- **Borda:** $u_1(\pi) = 5$; $u_2(\pi) = 5 + 4 = 9$; $u_3(\pi) = 4 + 3 = 7$.
- **Lexicographic:** $u_1(\pi) = 16$; $u_2(\pi) = 24$; $u_3(\pi) = 12$.
- **s-QI:** $u_1(\pi) = 1 + s_1/M$; $u_2(\pi) = 2 + s_1+s_2/M$; $u_3(\pi) = 2 + s_2+s_3/M$.
- **2-approval:** $u_1(\pi) = 1$; $u_2(\pi) = 2$; $u_3(\pi) = 1$.

Scoring allocation rules

Back to our resource allocation problem...

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Interpretation:
Borda SF
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Quasi-Indifference SF
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k-Approval SF	

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Lexicographic SF		
Quasi-Indifference SF		
k-Approval SF		

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$\rightsquigarrow$ 12 **positional scoring allocation rules**

How to interpret scores?

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Two interpretations of scores:

1. Compromise between all agents (domain knowledge, learned, ...)
2. Perception of the center

Properties

1. **Separability:** Violated by almost all our rules
2. **Monotonicity:** Satisfied by all our rules
3. **Global monotonicity:** Violated by almost all our rules
   with **strictly decreasing scoring vector**
4. **Pos. object monotonicity:** Satisfied by $+$ for $n = 2$, but **violated**
   for $n \geq 3$ and **strictly decreasing scoring vector**

Baumeister, D., Bouveret, S., Lang, J., Nguyen, N., Nguyen, T., and Rothe, J. (2014).
Scoring rules for the allocation of indivisible goods.
In *Proceedings of ECAI'14*, pages 75–80. IOS Press.

Complexity

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For each pair (scoring vector, social criterion), what is the complexity of...

1. **Optimal Allocation Value (OAV):** is it possible to find an allocation of social welfare $\geq K$?
2. **Optimal Allocation (OA):** does π belong to the set of optimal allocations?
3. **Find Optimal Allocation (FOA):** find an optimal allocation.

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Most results for min carry over to leximin.

Approximation

Most cases are **hard**...

Question: *Is it possible to efficiently compute* ***good*** *(but potentially suboptimal) allocations?*

Approximation

Most cases are **hard**...

Question: *Is it possible to efficiently compute* ***good*** *(but potentially suboptimal) allocations?*

Our approach: Instead of giving general approximation results, we:

- focus on a **simple** allocation protocol [Bouveret and Lang, 2011];
- and try to analyze how good the allocations it gives are.

Bouveret, S. and Lang, J. (2011).
A general elicitation-free protocol for allocating indivisible goods.
In *Proceedings of IJCAI'11*, pages 73–78. IJCAI.

An elicitation-free protocol...

Ask the agents to pick in turn their most preferred object among the remaining ones, according to some **predefined sequence** σ.

Example

3 agents 1, 2, 3 / 6 objects / sequence 123321 $\rightarrow$ 1 chooses first (and takes her preferred object), then 2, then 3, then 3 again...

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Here we focus on **regular sequences** σ of the kind $(1 \dots n)^*$ and **Borda**.

Price of elicitation-freeness

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Multiplicative Price of Elicitation-Freeness:

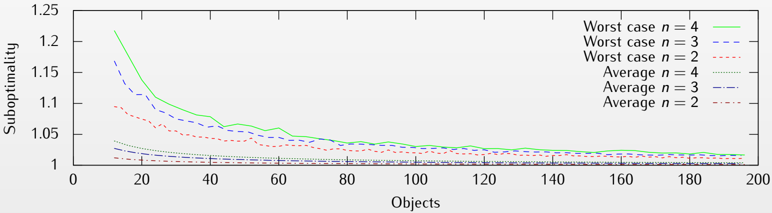
worst case ratio $sw^{\text{opt}}/sw(\sigma)$, for a sequence σ

Additive Price of Elicitation-Freeness:

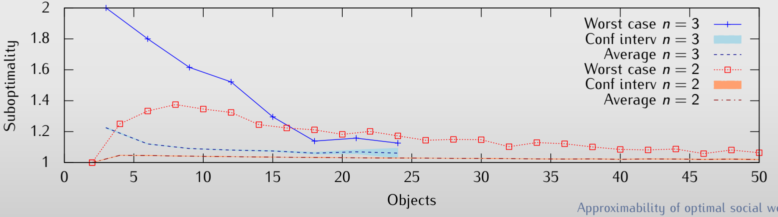
worst case difference $sw^{\text{opt}} - sw(\sigma)$, for a sequence σ

Experimental results

For **classical utilitarianism** ($\sum_i u_i(\pi)$):



For **egalitarianism** ($\min_i u_i(\pi)$):



Classical utilitarianism

For $m = kn$ objects,

$$MPEF_{+} \geq 1 + \frac{mn - m - n^2 + n}{m^2 + mn}$$

Why is this true?

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- $2 : o_5 \succ o_6 \succ o_1 \succ o_2 \succ o_3 \succ o_4$
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Corollary:
If $n = 2$ and $m = 2k$,

$$1 + \frac{m-2}{m(m+2)} \leq MPEF_{+} \leq \frac{3}{2} + \frac{3}{2m+2}$$

Egalitarianism

For $m = kn$ objects,

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- Upper bound $MPEF_{\text{min}}$ using best and worst case profile

Corollary:
If $n = 2$ and $m = 2k$,

$$MPEF_{\text{min}} \leq \frac{3}{2} + \frac{5}{m + 4}$$

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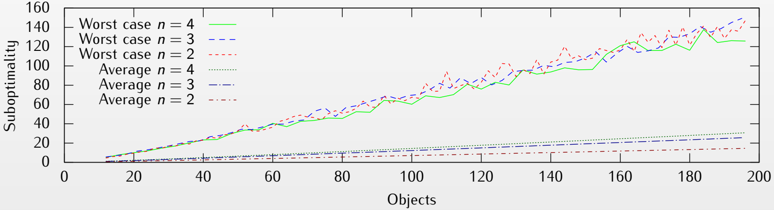

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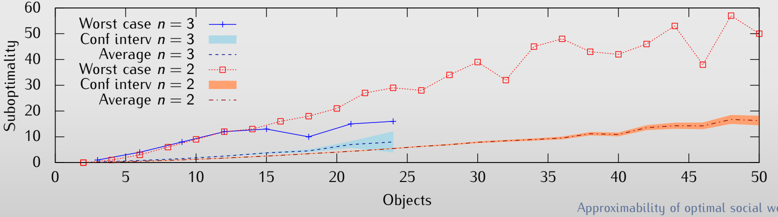

Formal bounds

For $m = kn$ objects,

$$\frac{(n-1)(m-n)}{2} \leq APEF_{+} \leq \frac{(m-n)(mn-m+n^2+n)}{2n}$$

$$APEF_{\min} \leq \frac{m^2n - mn - m^2 + mn^2}{2n^2}$$

Future Work

- Closing gaps (upper and lower bounds)
- Relationships (rank weighted utilitarianism, inequality indices)
- Exact characterizations, manipulation (scoring allocation rules)

Discussion

On finding rankings with approximately optimal Kemeny score:

> *An approximation algorithm for a voting rule is, in effect, a different voting rule; and in real-world elections, voters may feel deceived if a different voting rule is used than the one that was promised to them.*
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Does this argument hold in the resource allocation setting as well?

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