First International Conference
Modelling and Development of Intelligent Systems
Sibiu - Romania, 22-25 October, 2009

Departure model and its mathematical expression

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Abstract

For control of the traffic on traffic lights controlled intersections, it is very important to understand the behavior of the traffic flow. Particularly today it can be very difficult to regulate traffic networks, which are very complicated. Movement of vehicles on traffic lights controlled intersections is determined by many characteristics, which are essential for regulating or modelling of the traffic flow. Departure model is just one of these characteristics. It is about finding the time the vehicles need to leave the space in front of the stop line after the appearance of the green light. The resulting time is affected by many factors that are more or less relevant. Our aim is to create a new mathematical model, which can take into account these factors, find dependence between them and thus be more accurate for the specific intersections.

1 Introduction

If we want to examine behavior of a traffic flow and thus dynamics of vehicles´ movement on the traffic lights controlled intersection, it is necessary to divide the system into several parts (sub-systems) and to examine them separately. Traffic flow is like an organism, which is composed of heterogeneous parts. To better understand it, it is useful to work with its smaller parts. Movement of vehicles on a traffic lights controlled intersection can thus be divided into the following parts:

- arrival of a vehicle on the intersection;
- departure of a vehicle from the "pre-shift" area;
- movement of a vehicle inside the intersection;
- departure of a vehicle from the intersection.

Particularly the departure model helps us characterize dynamics of vehicles´ movement upon their departure from the "pre-shift" area. In simple terms, the departure model is determined by departure times (the so-called entry times), which help us determine the number of vehicles able to move through the intersection during the green light phase.

The first departure model was developed by Greenshilds ([7]) already in 1947. Since then, several other departure models were developed, for example by Professor Medelská ([6]) or by Webster. Each of them works with their own departure times, which are determined empirically. It is therefore probably not necessarily exact to refer to them as departure models (we will stay with this term however), but rather as to calculated average arrival times for a specific sequence of vehicles in a line. Professor Medelská attempted a mathematical departure model as well ([6]). She subsequently presented the measured and averaged entry times with the help of a quadratic equation.

Since the last (revised) departure model was developed in 1979, these models are not capable to adequately reflect current circumstances of road transportation.

This was fully confirmed by a research proposal of Ministry of Education, Youth, and Sports of the Czech Republic (MSM 21260025 – Modeling of Traffic Processes). Its main aim was to create such a simulation program, which would be capable of modeling different types of controlling an intersection. This simulation program allows for choosing of one of the known departure models. The program was naturally confronted with the real data obtained at an observed intersection and its outcome was biased particularly because it made use of departure models no longer corresponding to a reality.

Solution of this problem lies in closer examination of patterns in a departure model and in attempting to define own mathematical apparatus, which would better reflect contemporary traffic needs.

2 Departure model

Departure model is often defined by the values of entry times of vehicles entering the area of traffic light controlled intersection during the green light interval. In other words, the entry time of the specific vehicle in the row is the time required for passage of this vehicle (of its front part) through the "stop" line related to the beginning of the green light interval. We can thus say that the departure model describes behavior of vehicles standing in a row in front of a light controlled intersection and their subsequent movement at the green light until they cross the "stop" line.

Entry time can be divided into these parts:

- reaction of a driver to signal "go";
- time necessary for acceleration;
- time necessary for clear-out of the "pre-shift" area until the point when the vehicle did not yet cross the "stop" line with its front bumper.

As already stated, several authors created their own departure model and these models bear their names.

According to ([6]), departure of vehicles can be characterized by the following patterns:

- The second vehicle in the row needs most of the time, because the driver reacts only to the movement of the first vehicle.
- Vehicles from the fifth position onwards move at equal time intervals as long as a row of waiting vehicles exists.
- If there is no longer a row of waiting vehicles and there is still signal "go", vehicles enter the intersection randomly.

The following table illustrates the departure model (entry times) by different authors:

Vehicle sequence	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Each additional**
Greenschields	3.8	6.9	9.6	12.0	14.2	16.3	18.4	+ 2.1
Fischer	3.1	5.4	7.5	9.5	11.4	13.3	15.2	+ 1.9
Medelská	2.3	5.5	8.3	10.8	13.1	15.2	17.2	+ 2.0
Medelská (revised)	1.2	3.9	6.6	9.1	11.6	14.0	16.3	+ 2.3
ÚSMD	3.6	6.5	8.9	11.2	13.4	15.4	17.4	+ 2.0
Webster	1	2	3	4	5	6	7	+ 1.0

Table 1: Departure models of different authors

This means that for example according to Fischer, vehicle, which is the fourth in the line, crosses the intersection after the signal "go" comes up in 9.5 sec.

3 Mathematical expression of our departure model

We assume the following conditions for our departure model. Departure of vehicles in a row is dependent on the departure of the previous vehicle (naturally with the exception of the first one), as well as on parameters, which influence its departure. We will consider the model in the following equation form:

$$y_t = \beta u_t + \kappa + \epsilon_t \tag{1}$$

where,

- $y_t = [y_{1;t}, y_{2;t}, \dots, y_{m;t}]'$ is the vector of modeled departure times of individual vehicles in the row where its length is expressed by m and the time index by t
- β is matrix of parameters
  - $\beta = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{bmatrix}$
  - n is the number of parameters
- $u = [u_{1;t}, u_{2;t}, \dots, u_{n;t}]'$ is the vector of values of variables influencing departures
- $\kappa = [\kappa_1, \kappa_2, \dots, \kappa_m]'$ is the model´s constant
- $\epsilon = [\epsilon_1, \epsilon_2, \dots, \epsilon_m]'$ is the noise with the median value of zero and covariance matrix R

Covariance matrix R is symmetric and positive definite and can be therefore unequivocally divided into form "LDL‘", where "L" is the lower rectangular matrix with numbers "1" on the main diagonal "D" and "D" is diagonal matrix.

$$R = LDL' = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \rho_{21} & 1 & 0 & \cdots & 0 \\ \rho_{31} & \rho_{32} & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \rho_{m1} & \rho_{m2} & \cdots & \rho_{mn-1} & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3^2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \sigma_m^2 \end{bmatrix} \begin{bmatrix} 1 & \rho_{21} & \rho_{31} & \cdots & \rho_{m1} \\ 0 & 1 & \rho_{32} & \cdots & \rho_{m2} \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \rho_{mn-1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Now, instead of the original noise ϵ_t, we will consider a new noise e_t, in such a way that $\epsilon_t = Le_t$ and noise has e_t a diagonal covariance matrix "D". It holds that:

$$E[\epsilon_t \epsilon_t'] = E[Le_t e_t' L'] = LE[e_t e_t']L' = R$$

Using an established substitution, it is possible to record the model of Equation 1 in this form

$$y_t = bu_t + k + Le_t$$

If we multiply the entire previous equation from the left by the matrix L^{-1}, which always exists and which is also lower triangular with numbers "1" on the main diagonal, we arrive at

$$L^{-1} y_t = L^{-1} \beta u_t + L^{-1} \kappa + e_t$$

If we create a substitution $a = -L^{-1} + I$, where

- I is a unit matrix or order m, then it is possible to record this model in the form

$$y_t = ay_t + bu_t + k + e_t \tag{2}$$

Whereas matrix a has on and above the main diagonal zeros, elements below the main diagonal are marked $a_{i,j}$. Noise e_t has elements, which are not mutually correlated. We can record the model as m of independent equations in the following manner.

$$y_{1;t} = \beta_{11} u_{1;t} + \beta_{12} u_{2;t} + \cdots + \beta_{1n} u_{n;t} + e_{1;t}$$
$$y_{2;t} = a_{21} y_{1;t} + \beta_{21} u_{1;t} + \beta_{22} u_{2;t} + \cdots + \beta_{2n} u_{n;t} + r_{21} e_{1;t} + e_{2;t}$$
$$y_{3;t} = a_{31} y_{1;t} + a_{32} y_{2;t} + \beta_{31} u_{1;t} + \beta_{32} u_{2;t} + \cdots + r_{31} e_{1;t} + r_{32} e_{2;t} + e_{3;t}$$

$$y_{m;t} = a_{m1}y_{1;t} + a_{m2}y_{2;t} + \cdots + a_{mm-1}y_{m-1;t} + \beta_{m1}u_{1;t} + \beta_{m2}u_{2;t} + \cdots + \beta_{mm-1}u_{m-1;t} + r_{m1}e_{1;t} + r_{m2}e_{2;t} + \cdots + r_{mm-1}e_{m-1;t} + e_{m;t}$$

where

- $e_{i;t}$ have zero expectations and dispersions σ_i^2.

This is the matrix expression of this model:

$$\begin{bmatrix} y_{1;t} \\ y_{2;t} \\ y_{3;t} \\ \vdots \\ y_{m;t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{m1} & a_{m2} & \cdots & a_{m(m-1)} & 0 \end{bmatrix} \begin{bmatrix} y_{1;t} \\ y_{2;t} \\ y_{3;t} \\ \vdots \\ y_{m;t} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ b_{31} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \begin{bmatrix} u_{1;t} \\ u_{2;t} \\ u_{3;t} \\ \vdots \\ u_{n;t} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ r_{21} & 1 & 0 & \cdots & 0 \\ r_{31} & r_{32} & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ r_{m1} & r_{m2} & \cdots & r_{m(m-1)} & 1 \end{bmatrix} \begin{bmatrix} e_{1;t} \\ e_{2;t} \\ e_{3;t} \\ \vdots \\ e_{m;t} \end{bmatrix} \quad (3)$$

In equation 2 or 3, we arrive precisely at the mathematical application which we need. We model the departure of a vehicle, or rather its entry time, with dependence on departure of previous vehicles and other parameters which influence it.

4 Parameters influencing the departure model

It is apparent that entry times of vehicles are, besides the preceding vehicle, influenced by other factors, or parameters, which can vary for different vehicles. It is difficult, if not impossible, to define these parameters, because of their number and our capacity to measure them. We also need to consider whether or not the particular parameter indeed influences the entry time of the vehicle. This is very important, because it helps us to develop a suitable complete model. Among the measured parameters are, for example: type of the vehicle, weather, road surface, descent angle, geometrical order of the intersection, impact of vehicles in the opposite direction, or hypothetically whether the driver is male or female. We may also not forget those parameters, which are difficult to measure, such as health, age, psychological well-being, or alertness of the driver.

5 Real data and our departure model

We tested our model on a total of 2,254 vehicles. This examination is only a first one of many and data is still being collected. Nevertheless, even this preliminary evaluation reveals that the departure model, which we propose, differs from the other ones mentioned. Figure 1 displays comparison of our departure model with other models vis-à-vis the measured data. This comparison was undertaken using the method of least squares.

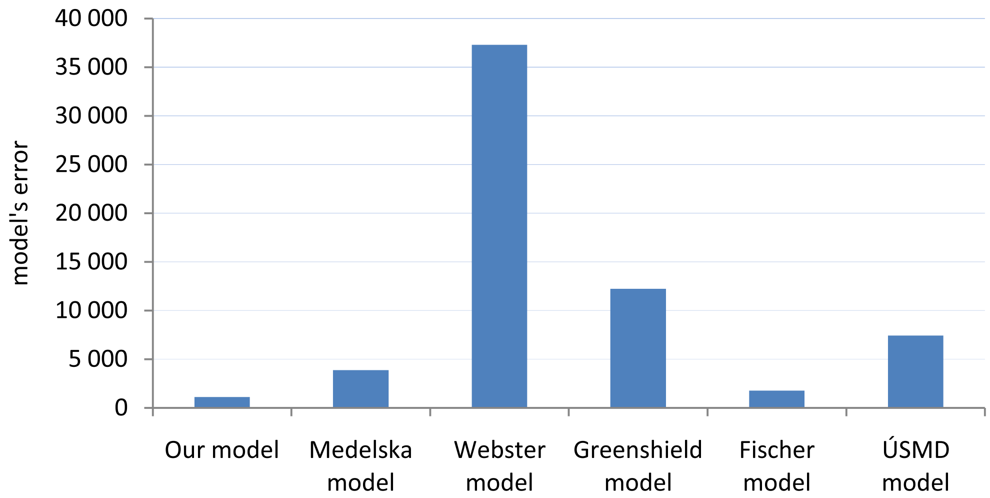

Fig. 1: Comparison of Departure models

If we compare our model with the departure model of Fischer, which shows the second smallest error value, we arrive at the following result: the model of Fischer is worse by approximately 62%, or rather the sum total of errors of the model of Fischer is worse than our departure model by 62%. If we compare our departure model with the one of Professor Medelská (revised), whose margin of error is the third smallest, we find that her departure model is worse than ours by 252%.

6 Conclusion

According to the preliminary results (see Figure), our departure model is significantly more accurate than other models. For example, the second most accurate model of Fischer, is worse than ours by 62%. The mathematical apparatus which we have created is adjusted as to incorporate into the departure model different parameters. It is thus very strong, because these parameters can render our departure model substantially more accurate. Based on different parameters we can then define departure model for different specific intersection, or, even better, for all intersections with identical parameters. Such variables must not be only physical (such as angle of the road), but also psychological. These are, however, difficult to measure. Queries, examination, and verification of these variables, which influence the departure of vehicles queuing in front of the light controlled intersection, is demanding and thus requires further research.

Acknowledgement: This paper is founded from the research project Czech Ministry of Education No. MSM 6840770043.

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