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On the approximate solution of a functional-integral equation

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Abstract

In this paper we consider the following functional-integral equation with linear modifications of the arguments:

$$u(x,y) = \int_0^x \int_0^y K(s,t,u(s,t),u(\lambda s,t),u(s,\mu t))dsdt, (x,y) \in [0,a]\times[0,b],$$

where $0<\lambda<1, 0<\mu<1, \quad K \in C([0,a]\times[0,b]\times\mathbb{R}^3)$.

Using the Picard operators' technique we obtain existence and uniqueness results for the solution of this equation.

By applying the successive approximations method and by using a cubature formula we give an algorithm for the approximate solution.

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1 Introduction

Many problems from astronomy, chemistry, biology, economics, engineering lead to mathematical models described by functional-integral equations.

The theory of functional-integral equations has been developed in the last fifty years. Many monographs appeared: Bellman and Cooke [3] (1963), Halanay [14] (1965), Elsgoltz and Norkin [9] (1971), Bernfeld and Lakshmikantham [4] (1974), Hale [12] (1977), Lakshmikantham [19] (1984), Azbelev, Maksimov and Rahmatulina [1] (1991), Corduneanu [7] (1991), Gopalsamy [10] (1992), Hale and Verduyn Lunel [13] (1993), Guo, Lakshmikantham and Liu [11] (1996) such as a large number of papers. We quote here [2], [5], [15], [25], [31], [33].

A special class is represented by the integral equations with affine modifications of the arguments, which can be with delay or with linear modifications of the arguments. The latter equations have been developed in connection with the pantograph equation $x'(t) = ax(\lambda t)$ and in connection with problems for the mentioned equation and for some of its generalizations (see [17], [18], [24], [32]).

Various Darboux-Ionescu problems for some equations with deviating arguments were presented by I.A. Rus in [27]. These problems are equivalent to some functional -integral equations.

In this paper we consider the following functional-integral equation with linear modifications of the arguments:

$$u(x,y) = \int_0^x \int_0^y K(s,t,u(s,t),u(\lambda s,t),u(s,\mu t))dsdt, (x,y) \in [0,a]\times[0,b],$$

where $0 < \lambda < 1, 0 < \mu < 1, \quad K \in C([0,a] \times [0,b] \times \mathbb{R}^3)$.

Using the Picard operators' technique (see I.A.Rus [29]), we obtain existence and uniqueness results for the solution of this equation.

By applying the successive approximations method and by using a cubature formula (see D.V. Ionescu [16]) we give an algorithm for the approximate solution.

2 Existence and uniqueness of the solution

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by
$F_A := \{x \in X | A(x) = x\}$ - the fixed point set of A;
$A^0 := 1_X, A^1 := A, A^{n+1} := A \circ A^n, \ n \in \mathbb{N}$.

Definition 1 *(Rus [29]) The operator A is a* ***Picard operator*** *if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;

(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.

Definition 2 *(Rus [29]) The operator A is a* ***weakly Picard operator*** *if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and its limit (which may depend on x_0) is a fixed point of A.*

Remark 1 *If the operator A is a weakly Picard operator and $F_A = \{x^*\}$, then A is a Picard operator.*

Now, we consider the functional-integral equation:

$$u(x,y) = \int_0^x \int_0^y K(s,t,u(s,t),u(\lambda s,t),u(s,\mu t))dsdt, (x,y) \in [0,a] \times [0,b], \quad (2.1)$$

where $0 < \lambda < 1, 0 < \mu < 1, \quad K \in C([0,a] \times [0,b] \times \mathbb{R}^3)$.

We consider the Banach space $(C[0,a] \times [0,b], ||.||_B)$, where

$$||u||_B = \max_{(x,y)\in[0,a]\times[0,b]} |u(x,y)|e^{-\tau(x+y)}, \tau \in \mathbb{R}_+,$$

and the operator $A : (C[0,a] \times [0,b], ||.||_B) \to (C[0,a] \times [0,b], ||.||_B)$, defined by

$$A(u)(x,y) := \int_0^x \int_0^y K(s,t,u(s,t),u(\lambda s,t),u(s,\mu t))dsdt.$$

We can write the equation (2.1) as a fixed point problem of the form: $u = A(u)$.

We have

Theorem 1 *We suppose that:*

(i) $K \in C([0,a] \times [0,b] \times \mathbb{R}^3)$;

(ii) there exists $L > 0$ such that

$$|K(s,t,u_1,v_1,w_1) - K(s,t,u_2,v_2,w_2)| \leq L(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|),$$

for all $(s,t) \in [0,a] \times [0,b]$ and all $u_i, v_i, w_i \in \mathbb{R}, i = 1,2$.

Then the equation (2.1) has a unique solution in $C([0,a] \times [0,b])$ and this solution can be obtained by the succesive approximation method, starting from any $u_0 \in C([0,a] \times [0,b])$.

Proof. We have

$$|A(u)(x,y)-A(v)(x,y)|\leq$$

$$\leq L(\int_0^x\int_0^y(|u(s,t)-v(s,t)|e^{-\tau(s+t)}e^{\tau(s+t)}dsdt+$$

$$+\int_0^x\int_0^y(|u(\lambda s,t)-v(\lambda s,t)|e^{-\tau(\lambda s+t)}e^{\tau(\lambda s+t)}dsdt+$$

$$+\int_0^x\int_0^y(|u(s,\mu t)-v(s,\mu t)|e^{-\tau(s+\mu t)}e^{\tau(s+\mu t)}dsdt)\leq$$

$$\leq \frac{L}{\tau^2}(1+\frac{1}{\lambda^2}+\frac{1}{\mu^2})e^{\tau(x+y)}||u-v||_B.$$

Therefore,

$$|A(u)(x,y)-A(v)(x,y)|e^{-\tau(x+y)}\leq \frac{L}{\tau^2}(1+\frac{1}{\lambda^2}+\frac{1}{\mu^2})||u-v||_B$$

for all $(x,y)\in[0,a]\times[0,b]$.

It follows that

$$||A(u)-A(v)||_B\leq \frac{L}{\tau^2}(1+\frac{1}{\lambda^2}+\frac{1}{\mu^2})||u-v||_B,$$

for all $u,v\in C([0,a]\times[0,b])$.

By choosing $\tau\in\mathbb{R}_+$ large enough, we have that A is a contraction. So A is a Picard operator. □

Now, we are looking for the solution of (2.1) in the following set

$$\begin{aligned}
Y &= \{u\in C^2([0,a]\times[0,b],J)|||u||_C\leq R_1,||\frac{\partial u}{\partial x}||_C\leq R_2,||\frac{\partial u}{\partial y}||_C\leq R_3,\\
||\frac{\partial^2 u}{\partial x^2}||_C &\leq R_4,||\frac{\partial^2 u}{\partial x\partial y}||_C\leq R_5,||\frac{\partial^2 u}{\partial y^2}||_C\leq R_6,\\
R_i &> 0, i=\overline{1,6}, J=[-r,r], r>0\}.
\end{aligned}$$

Here $||.||_C$ is the Tchebyschev norm.

Consider the Banach space $(Y,||.||_B)$ and the operator $C:(Y,||.||_B)\to(C^2([0,a]\times[0,b],J),||.||_B)$ defined by

$$C(u)(x,y):=\int_0^x\int_0^y K(s,t,u(s,t),u(\lambda s,t),u(s,\mu t))dsdt,$$

where $K\in C^2([0,a]\times[0,b]\times J^3)$.

We denote

$$M=\max_{[0,a]\times[0,b]\times J^3,|\beta|\leq 2}|\frac{\partial^{|\beta|}K}{\partial s^{\beta_1}\partial t^{\beta_2}\partial u^{\beta_3}\partial v^{\beta_4}\partial w^{\beta_5}}|.$$

Therefore

$$||C(u)(x,y)||_C\leq Mab,||\frac{\partial}{\partial x}C(u)(x,y)||_C\leq Mb,||\frac{\partial}{\partial y}C(u)(x,y)||_C\leq Ma,$$

$$||\frac{\partial^2}{\partial x^2}C(u)(x,y)||_C\leq Mb[1+R_2(\lambda+2)],||\frac{\partial^2}{\partial y^2}C(u)(x,y)||_C\leq Ma[1+R_3(\lambda+2)],$$

$$||\frac{\partial^2}{\partial x\partial y}C(u)(x,y)||_C\leq M.$$

We have

Theorem 2 *We suppose that*
(i) $K \in C^2([0,a] \times [0,b] \times J^3)$;
(ii) there exists $L > 0$ *such that*

$$|K(s,t,u_1,v_1,w_1) - K(s,t,u_2,v_2,w_2)| \leq L(|u_1-u_2| + |v_1-v_2| + |w_1-w_2|),$$

for all $(s,t) \in [0,a] \times [0,b]$ *and all* $u_i, v_i, w_i \in J, i = 1,2.$
(iii) $Mab \leq R_1$, $Mb \leq R_2$, $Ma \leq R_3$, $Mb[1+R_2(\lambda+2)] \leq R_4$, $M \leq R_5$, $Ma[1+R_3(\lambda+2)] \leq R_6$.
Then the functional-integral equation (2.1) *has a unique solution in* Y.

Proof. The equation (2.1) can be written as a fixed point problem $u = C(u)$. The condition *(iii)* insures us that Y is an invariant subset for the operator C. Similarly as above, by using *(ii)* we obtain that C is a Picard operator. □

3 The approximate solution

We give an algorithm for the approximate solution of the equation (2.1).

We suppose that the conditions in Theorem 5 are satisfied.

Let $u^* \in Y$ be the unique solution of this equation. This solution can be obtained by the succesive approximations method starting from any $u_0 \in Y$. Consider $u_0(x,y) = u_0$, where $u_0 \in \mathbb{R}$. Then

$$u_1(x,y) := \int_0^x \int_0^y K(s,t,u_0,u_0,u_0)dsdt, (x,y) \in [0,a] \times [0,b];$$

$$u_2(x,y) := \int_0^x \int_0^y K(s,t,u_1(s,t),u_1(\lambda s,t),u_1(s,\mu t))dsdt, (x,y) \in [0,a] \times [0,b];$$

....

$$\begin{array}{rcl} u_n(x,y) & : & = \int_0^x \int_0^y K(s,t,u_{n-1}(s,t),u_{n-1}(\lambda s,t),u_{n-1}(s,\mu t))dsdt, \\ (x,y) & \in & [0,a] \times [0,b]; \end{array}$$

....

We use the cubature formula (see D.V. Ionescu [16])

$$\begin{aligned} \int_0^a \int_0^b f(x,y)dxdy &= \frac{ba}{2pq}[\sum_{j=0}^{p-1} f(x_j,0) + \sum_{j=1}^{p} f(x_j,b) + \sum_{l=1}^{q-1} f(0,y_l) + \\ &+ \sum_{l=1}^{q-1} f(a,y_l) + 2\sum_{j=1}^{p-1}\sum_{l=1}^{q-1} f(x_j,y_l)] + R_f. \qquad (3.1) \end{aligned}$$

An upper bound for the remainder R_f is given by

$$|R_f| \leq \frac{ab}{12}(\frac{a^2}{p^2} + 3\frac{ab}{pq} + \frac{b^2}{q^2})M_2,$$

where

$$M_2 = \max_{[0,a]\times[0,b]} \left\{ |\frac{\partial^2 f}{\partial x^2}|, |\frac{\partial^2 f}{\partial x \partial y}|, |\frac{\partial^2 f}{\partial y^2}| \right\}.$$

Here

$$\begin{aligned} 0 &< x_1 < ... < x_{j-1} < x_j < ... < x_{p-1} < a, \\ 0 &< y_1 < ... < y_{l-1} < y_l < ... < y_{q-1} < b, \end{aligned}$$

and $x_j = \frac{a}{p}j, j = \overline{1,p}, y_l = \frac{b}{q}l, l = \overline{1,q}$.
We have

Theorem 3 *We suppose that all the conditions in Theorem 5 are satisfied. The values of the succesive approximations sequence on the knots* $(x_i, y_k) \in [0,a] \times [0,b]$, $i = \overline{0,p}, k = \overline{0,q}$ *are*

$$\begin{aligned} u_n(x_i, y_k) &= \frac{ba}{2pq}[\sum_{j=0}^{i-1} K(x_j, 0, u_{n-1}(x_j, 0), u_{n-1}(\lambda x_j, 0), u_{n-1}(x_j, 0)) + \\ &+ \sum_{j=1}^{i} K(x_j, y_k, u_{n-1}(x_j, y_k), u_{n-1}(\lambda x_j, y_k), u_{n-1}(x_j, \mu y_k)) + \\ &+ \sum_{l=1}^{k-1} K(0, y_l, u_{n-1}(0, y_l), u_{n-1}(0, y_l), u_{n-1}(0, \mu y_l)) + \\ &+ \sum_{l=1}^{k-1} K(a, y_l, u_{n-1}(a, y_l), u_{n-1}(\lambda a, y_l), u_{n-1}(a, \mu y_l)) + \\ &+ 2\sum_{j=1}^{i-1}\sum_{l=1}^{k-1} K(x_j, y_l, u_{n-1}(x_j, y_l), u_{n-1}(\lambda x_j, y_l), u_{n-1}(x_j, \mu y_l))] + \\ &+ R_{n,i,k}, \qquad (3.2) \end{aligned}$$

where $i = \overline{1,p}$, $k = \overline{1,q}$, $n \in \mathbb{N}^*$, *and*

$$|R_{n,i,k}| \le \frac{ab}{12}(\frac{a^2}{i^2} + 3\frac{ab}{ik} + \frac{b^2}{k^2})M_0,$$

where M_0 *is a constant not depending on* n.

Proof. We have

$$\begin{aligned} u_n(x_i, y_k) &= \int_0^{x_i}\int_0^{y_k} K(s, t, u_{n-1}(s,t), u_{n-1}(\lambda s, t), u_{n-1}(s, \mu t))dsdt, \\ i &= \overline{0,p}, k = \overline{0,q}, n \in \mathbb{N}^*. \end{aligned}$$

By using the cubature formula (3.1), we obtain (3.2).
For $x_m \le \lambda x_j < x_{m+1}$, we consider
$u_{n-1}(\lambda x_j, y_k) := u_{n-1}(x_m, y_k), m = \overline{0, i-1}$, and for $y_r \le \mu y_l < y_{r+1}$, we consider
$u_{n-1}(x_j, \mu y_l) := u_{n-1}(x_j, y_r), r = \overline{0, k-1}$.
Here

$$|R_{n,i,k}| \le \frac{x_i y_k}{12}(\frac{x_i^2}{i^2} + 3\frac{x_i y_k}{ik} + \frac{y_k^2}{k^2})M_{2,n,i,k}$$

and

$$M_{2,n,i,k} = \max_{[0,x_i]\times[0,y_k]}\left\{|\frac{\partial^2 K_n}{\partial s^2}|, |\frac{\partial^2 K_n}{\partial s \partial t}|, |\frac{\partial^2 K_n}{\partial t^2}|\right\},$$

where,

$$K_n(s,t) := K(s, t, u_{n-1}(s,t), u_{n-1}(\lambda s, t), u_{n-1}(s, \mu t)).$$

We denote by $u_{n-1}(s,t)=\alpha, u_{n-1}(\lambda s,t)=\beta,$ and $u_{n-1}(s,\mu t)=\gamma$. So we obtain

$$\frac{\partial K_n}{\partial s}(s,t,\alpha,\beta,\gamma)=\frac{\partial K}{\partial s}+\frac{\partial K}{\partial \alpha}\frac{\partial u_{n-1}}{\partial s}(s,t)+\lambda\frac{\partial K}{\partial \beta}\frac{\partial u_{n-1}}{\partial s}(\lambda s,t)+\frac{\partial K}{\partial \gamma}\frac{\partial u_{n-1}}{\partial s}(s,\mu t);$$

$$\frac{\partial^2 K_n}{\partial s^2}(s,t,\alpha,\beta,\gamma)=\frac{\partial^2 K}{\partial s^2}+\frac{\partial^2 K}{\partial s\partial\alpha}\frac{\partial u_{n-1}}{\partial s}(s,t)+$$
$$+\lambda\frac{\partial^2 K}{\partial s\partial\beta}\frac{\partial u_{n-1}}{\partial s}(\lambda s,t)+\frac{\partial^2 K}{\partial s\partial\gamma}\frac{\partial u_{n-1}}{\partial s}(s,\mu t)+$$
$$+(\frac{\partial^2 K}{\partial s\partial\alpha}+\frac{\partial^2 K}{\partial \alpha^2}\frac{\partial u_{n-1}}{\partial s}(s,t)+\lambda\frac{\partial^2 K}{\partial\alpha\partial\beta}\frac{\partial u_{n-1}}{\partial s}(\lambda s,t)+$$
$$+\frac{\partial^2 K}{\partial\alpha\partial\gamma}\frac{\partial u_{n-1}}{\partial s}(s,\mu t))\frac{\partial u_{n-1}}{\partial s}(s,t)+\frac{\partial K}{\partial\alpha}\frac{\partial^2 u_{n-1}}{\partial s^2}(s,t)+$$
$$+\lambda[\frac{\partial^2 K}{\partial s\partial\beta}+\frac{\partial^2 K}{\partial\alpha\partial\beta}\frac{\partial u_{n-1}}{\partial s}(s,t)+\lambda\frac{\partial^2 K}{\partial\beta^2}\frac{\partial u_{n-1}}{\partial s}(\lambda s,t)+$$

$$+\frac{\partial^2 K}{\partial\gamma\partial\beta}\frac{\partial u_{n-1}}{\partial s}(s,\mu t)]\frac{\partial u_{n-1}}{\partial s}(\lambda s,t)+\lambda\frac{\partial K}{\partial\beta}\frac{\partial^2 u_{n-1}}{\partial s^2}(\lambda s,t)+$$

$$+[\frac{\partial^2 K}{\partial s\partial\gamma}+\frac{\partial^2 K}{\partial\alpha\partial\gamma}\frac{\partial u_{n-1}}{\partial s}(s,t)+\lambda\frac{\partial^2 K}{\partial\beta\partial\gamma}\frac{\partial u_{n-1}}{\partial s}(\lambda s,t)+$$
$$+\frac{\partial^2 K}{\partial\gamma^2}\frac{\partial u_{n-1}}{\partial s}(s,\mu t)]\frac{\partial u_{n-1}}{\partial s}(s,\mu t)+\frac{\partial K}{\partial\gamma}\frac{\partial^2 u_{n-1}}{\partial s^2}(s,\mu t).$$

Because

$$\alpha=u_{n-1}(s,t)=\int_0^s\int_0^t K(s,t,u_{n-2}(s,t),u_{n-2}(\lambda s,t),u_{n-2}(s,\mu t))dsdt,$$

we have

$$\frac{\partial\alpha}{\partial s}=\int_0^t K(s,t,u_{n-2}(s,t),u_{n-2}(\lambda s,t),u_{n-2}(s,\mu t))dt,$$

$$\frac{\partial^2\alpha}{\partial s^2}=\int_0^t(\frac{\partial K}{\partial s}+\frac{\partial K}{\partial\alpha}\frac{\partial u_{n-2}}{\partial s}(s,t)+\lambda\frac{\partial K}{\partial\beta}\frac{\partial u_{n-2}}{\partial s}(\lambda s,t)+\frac{\partial K}{\partial\gamma}\frac{\partial u_{n-2}}{\partial s}(s,\mu t))dt,$$

and

$$|\frac{\partial\alpha}{\partial s}|\leq Mb, |\frac{\partial^2\alpha}{\partial s^2}|\leq Mb[1+Mb(\lambda+2)].$$

It follows that

$$|\frac{\partial^2 K_n}{\partial s^2}|\leq M+M^2b+\lambda M^2b+M^2b+(M+M^2b+\lambda M^2b+M^2b)Mb+$$
$$+M^2b(1+\lambda Mb+2Mb)+\lambda(M+M^2b+\lambda M^2b+M^2b)Mb+$$
$$+\lambda M(Mb+\lambda M^2b^2+2M^2b^2)+(M+M^2b+\lambda M^2b+M^2b)Mb+$$
$$+M^2b(1+\lambda Mb+2Mb)=M_1.$$

Similarly as above we have

$$|\frac{\partial^2 K_n}{\partial t^2}|\leq M_2 \quad and \quad |\frac{\partial^2 K_n}{\partial s\partial t}|\leq M_3.$$

We choose

$$M_0=\max\{M_1,M_2,M_3\}.$$

□

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