First International Conference
Modelling and Development of Intelligent Systems
Sibiu - Romania, 22-25 October, 2009

On some linear positive operators: statistical approximation and q-generalizations

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Abstract

This paper is focused on sequences of linear positive operators, the starting point being represented by Popoviciu-Bohman-Korovkin criterion. Our first aim is to sum up recent investigation on statistical convergence of this type of approximation processes. The second aim is to construct a bivariate extension of Stancu discrete operators. This generalization is based on q-integers and on the tensor product method.

1 Introduction

Let $(L_n)_{n\geq 1}$ be a sequence of linear positive operators acting on the space $C([a,b])$ of all real-valued and continuous functions defined on the interval $[a,b]$, equipped with the norm $\|\cdot\|$ of the uniform convergence. Popoviciu-Bohman-Korovkin's theorem asserts: if the operators L_n, $n \in \mathbb{N}$, map $C([a,b])$ into itself such that $\lim_n \|L_n e_j - e_j\| = 0$ for $j = 0,1,2$, then one has $\lim_n \|L_n f - f\| = 0$ for every $f \in C([a,b])$. Here e_j represents the monomial of j-th degree, $j \in \{0,1,2\}$. A special extension of this criterion consists in replacing the uniform convergence by statistical convergence. This approach models and improves the technique of signals approximation in different function spaces. This is useful both in various areas of functional analysis and in obtaining numerical solutions of some differential and integral equations. Following this direction, recent results regarding statistical convergence criterions are collected in Section 2. Further on, we bring into light a bi-dimensional linear positive operator of Bernstein-Stancu type based on Quantum Calculus. The construction technique takes one's stand on the tensor product.

2 Statistical convergence theorems of Korovkin type

First of all, we briefly recall some basic facts with regard to the notion of statistical convergence. This concept, originally appeared in Steinhaus [22] and Fast [8] papers, is based on the notion of the density of subsets of $\mathbb{N} = \{1,2,3,\dots\}$ and it can be viewed as a regular method of summability of sequences.

The density of a set $K \subset \mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k),$$

provided the limit exists, where χ_K is the characteristic function of K. Clearly, the sum of the right hand side represents the cardinality of the set $\{k \leq n : \ k \in K\}$. A sequence $x = (x_k)_{k\geq 1}$ is statistically convergent to a real number L, denoted $st - \lim_k x_k = L$, if, for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}) = 0$$

holds. Closely related to this notion is A-statistical convergence, where $A = (a_{n,k})$ is an infinite summability matrix. For a given sequence $x = (x_k)_{k\geq 1}$ the A-transform of x, denoted by $Ax = ((Ax)_n)$, is defined as follows

$$(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n \in \mathbb{N},$$

provided the series converges for each n.

Suppose that A is non-negative regular summability matrix. Recall: A-summation method is regular if, whenever the sequence x converges to L, $\lim_n (Ax)_n = L$. The sequence x is A-statistically convergent to the real number L if, for every $\varepsilon > 0$, one has

$$\lim_{n\to\infty} \sum_{k\in I(\varepsilon)} a_{n,k} = 0,$$

where $I(\varepsilon) = \{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}$. We denote this limit by $st_A - \lim_k x_k = L$.

Taking in view both the definition of A-statistical convergence and the three regularity conditions claimed by Silverman-Toeplitz theorem, we notice that every convergent sequence is A-statistically convergent.

In the particular case $A = C_1$, the Cesàro matrix of first order, A-statistical convergence reduces to statistical convergence, see, e.g., [8], [9]. Also, if A is the identity matrix, then A-statistical convergence coincides with the ordinary convergence.

In Approximation Theory by linear positive operators, the statistical convergence has been examined for the first time by A.D. Gadjiev and C. Orhan. Popoviciu-Bohman-Korovkin criterion via statistical convergence will be read as follows [12; Theorem 1].

Theorem 1 *If the sequence of positive linear operators $L_n : C([a,b]) \to B([a,b])$ satisfies the conditions*

$$st - \lim_n \|L_n e_j - e_j\| = 0, \quad j \in \{0,1,2\}, \tag{1}$$

then, for any function $f \in C([a,b])$, we have

$$st - \lim_n \|L_n f - f\| = 0.$$

As usual, $B([a,b])$ stands for the space of all real valued bounded functions defined on $[a,b]$, endowed with the sup-norm.

Remark 1 *Examining the proof of Theorem 1 given by the authors, we notice that this statement is also true for A-statistical convergence, where A is a non-negative regular summability matrix.*

The quoted paper was to mean the beginning of an intensive study developed by many researches for obtaining criterions in order to decide the statistical convergence of a given sequence of linear positive operators in various function spaces.

In what follows, we exhibit noteworthy results which certify the achievements in this research field.

Using the concept of the rate of statistical convergence, a general class of linear positive operators defined on the space $C([0,b])$, $0 < b < 1$, has been investigated by Doğru, Duman and Özarslan [18].

Further on, set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and let $e_{ij}(x,y) = x^i y^j$, $i \in \mathbb{N}_0$, $j \in \mathbb{N}_0$, $i + j \leq 2$, stand for the test functions corresponding to the bi-dimensional case of the classical criterion. The following A-statistical analogue of Korovkin-type approximation theorem in bi-dimensional case has been obtained by Erkuş and Duman [7; Theorem 2.1].

Theorem 2 *Let A be a non-negative regular summability matrix and let $(L_n)_n$ be a sequence of positive linear operators from $C(I \times J)$ into $C(I \times J)$, where $I = [a,b]$ and $J = [c,d]$. Then, for all $f \in C(I \times J)$,*

$$st_A - \lim_n \|L_n f - f\| = 0,$$

if and only if the following identities hold

$$\begin{cases} st_A - \lim_n \|L_n e_{i,j} - e_{i,j}\| = 0, \ (i,j) \in \{(0,0),(0,1),(1,0)\}, \\ st_A - \lim_n \|L_n(e_{2,0} + e_{0,2}) - (e_{2,0} + e_{0,2})\| = 0. \end{cases}$$

In the above, $\|\cdot\|$ indicates the sup-norm of the space $C(I \times J)$.

Remark 2 *In the Banach space $C(I \times J)$, if A-statistical convergence is replaced by the uniform convergence, then Theorem 2 turns into a result due to Volkov [23].*

Now, we make a halt in weighted spaces. A function $\rho \in \mathbb{R}^{\mathbb{R}}$ is usually called a *weight function* if it is continuous on the domain satisfying the conditions $\rho \geq e_0$ and $\lim_{|x|\to\infty} \rho(x) = \infty$. Let consider the spaces

$$\begin{aligned} &B_\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \text{ a constant } M_f \text{ depending on } f \text{ exists such that } |f| \leq M_f \rho\}, \\ &C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R}) \mid f \text{ continuous on } \mathbb{R}\}, \end{aligned}$$

endowed with the common norm $\|\cdot\|_\rho$, where $\|f\|_\rho = \sup_{x\in\mathbb{R}} \dfrac{|f(x)|}{\rho(x)}$. Duman and Orhan [6; Theorem 3] proved the following weighted Korovkin type theorem via A-statistical convergence.

Theorem 3 *Let A be a non-negative regular summability matrix and let ρ_1, ρ_2 be weight functions such that*

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0.$$

Assume that $(L_n)_{n\geq 1}$ is a sequence of positive linear operators acting from $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$. Then

$$st_A - \lim_n \|L_n f - f\|_{\rho_2} = 0 \text{ for all } f \in C_{\rho_1}(\mathbb{R}),$$

if and only if

$$st_A - \lim_n \|L_n F_j - F_j\|_{\rho_1} = 0, \quad j \in \{0,1,2\},$$

where $F_j(x) = x^j(1+x^2)^{-1}\rho_1(x)$, $j = 0,1,2$.

Remark 3 *As the authors of the above theorem specify, substituting $st_A - \lim_n$ by ordinary limit one reobtains a result established by Gadjiev [11].*

Aiming at the same approach, we present an abstract version of the Korovkin type approximation criterion. Let (X,d) be an arbitrary compact metric space. We denote by $C(X)$ the space of all real-valued functions continuous on X. In the sequel $\mathbf{1}$ denotes the constant function 1. On the basis of [1; Theorems 1, 2] we state.

Theorem 4 *Let L_n, $n \in \mathbb{N}$, be positive linear operators acting on $C(X)$, where (X,d) is a compact metric space. Let α_n, β_n, $n \in \mathbb{N}$, be defined as follows*

$$\alpha_n(x) = L_n(d(\cdot,x),x), \quad \beta_n(x) = L_n(d^2(\cdot,x),x), \quad x \in X.$$

a) If

$$st - \lim_n \|L_n(\mathbf{1}) - \mathbf{1}\| = 0 \quad \text{and} \quad st - \lim_n \|\alpha_n\| = 0, \tag{2}$$

then the following identity holds

$$st - \lim_n \|L_n f - f\| = 0 \text{ for all } f \in C(X). \tag{3}$$

b) If $L_n(\mathbf{1}) = \mathbf{1}$ and $st - \lim_n \|\beta_n\| = 0$, then (3) also holds.

We point out the following particular case. Let X be a real vector space endowed with an inner product $(\cdot,\cdot)$ and let Y be an arbitrary compact subset of X. We define the functions e, e_x, a_n, b_n $(n\in\mathbb{N})$ belonging to $\mathbb{R}^Y$ as follows

$$\begin{cases} e(x)=(x,x), & a_n(x)=(L_ne)(x)-e(x), \quad x\in Y,\\ e_x(t)=(x,t), & b_n(x)=(L_ne_x)(x)-e(x), \quad t\in Y, x\in Y.\end{cases}$$

Since $\beta_n(x)=a_n(x)-2b_n(x)+e(x)((L_n\mathbf{1})(x)-1)$, assuming

$$L_n(\mathbf{1})=1 \quad \text{and} \quad st-\lim_n\|a_n\|=st-\lim_n\|b_n\|=0,$$

we deduce that identity (3) holds for every $f\in\mathbb{R}^Y$.

Remark 4 *Let* $X=\prod_{i=1}^{p}[a_i,b_i]\subset\mathbb{R}^p$ *be the p-dimensional parallelepiped. It's known that* $\left\{\mathbf{1},pr_1,\dots,pr_p,\sum_{j=1}^{p}pr_j^2\right\}$ *represents a Korovkin subset in* $C(X)$*, where* pr_j*,* $1\le j\le p$*, are the canonical projections on* X*. Based on a classical result, see, e.g., the monograph [2, p. 245] for deciding if a sequence of linear positive operators* $(L_n)_n$*,* $L_n:C(X)\to C(X)$*,* $n\in\mathbb{N}$*, forms a classical approximation process, we need to evaluate* L_n *on these* $p+2$ *test-functions. For establishing the statistical convergence, we need to prove nothing else but identities (2).*

3 Modified Stancu operators in q-calculus

Due to the intensive development of q-Calculus, various generalizations of many classical approximation processes of positive type have emerged. The first researches have been achieved by A. Lupaş [15] in 1987 and by G.M. Phillips [19] in 1997 who proposed q-variants of the original Bernstein operators. While Lupaş' operators are given by rational functions, Phillips' operators are composed of polynomials, named at present time q-Bernstein polynomials. During the last decade, acquiring popularity, these new polynomials have been studied by many authors who obtained a great number of results related to various properties of these operators. Reviews of the results on the q-Bernstein polynomials along with an extensive bibliography on this matter is given in [17]. At the same time, integral extensions in q-Calculus of Bernstein operators have been investigated. We refer to the q-analogues of the Bernstein-Durrmeyer operators [5], [13] and of the Bernstein-Kantorovich operators [20].

A generalization of Bernstein operators with great potential in Approximation Theory was achieved by D.D. Stancu [21]. Recently, G. Nowak [16] introduced a q-analogue of Stancu's operators. The goal of this section is to investigate statistical approximation property of q-Stancu operators. We also introduce an extension of this class acting on the space of real valued functions defined on a rectangular domain.

At this stage we require some preliminary results concerning q-integers, see, e.g., [14]. In the sequel, for our purposes, we just assume that $q\in(0,1)$. It should be mentioned that many results relating to q-Bernstein polynomials aim at the case $0<q<1$. The reason is simple: under this assumption, q-Bernstein polynomials generate positive linear operators and this aspect is significantly used in investigation.

For any $n\in\mathbb{N}_0$, the q-integer $[n]_q$ and the q-factorial $[n]_q!$ are respectively defined by

$$[n]_q=\sum_{j=0}^{n-1}q^j, \quad [n]_q!=\prod_{j=1}^{n}[j]_q, \quad n\in\mathbb{N},$$

and $[0]_q=0$, $[0]_q!=1$. The q-binomial coefficients or Gaussian coefficients denoted by $\begin{bmatrix} n\\ k\end{bmatrix}_q$ are defined as follows

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k=0,1,\dots,n.$$

Obviously, for $q=1$ one has $[n]_1! = n!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$, the ordinary binomial coefficients.

For $f \in C([0,1])$, $\alpha \geq 0$ and each $n \in \mathbb{N}$, in [16] have been defined the operators

$$(B_n^{q,\alpha} f)(x) = \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0,1], \tag{4}$$

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(x+\alpha[i]_q)\prod_{s=0}^{n-1-k}(1-q^s x+\alpha[s]_q)}{\prod_{i=0}^{n-1}(1+\alpha[i]_q)}, \tag{5}$$

$0 \leq k \leq n$. Note, an empty product is taken to be equal to 1. This class contains as special cases the following well-known sequences.

i) For $\alpha = 0$, $B_n^{q,0} \equiv B_n^q$ represents q-Bernstein operator introduced by Phillips [19].

ii) For $\alpha = 0$ and $q = 1$, $B_n^{1,0} \equiv B_n$ is the classical Bernstein polynomial.

iii) For $q = 1$, $B_n^{1,\alpha} \equiv B_n^{\langle \alpha \rangle}$ turns into Stancu operator [21] defined as follows

$$(B_n^{\langle \alpha \rangle} f)(x) = \frac{1}{1^{[n,-\alpha]}} \sum_{k=0}^{n} \binom{n}{k} x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]} f\left(\frac{k}{n}\right),$$

$x \in [0,1]$. Here $t^{[m,a]} = \prod_{j=0}^{m-1}(t-ja)$ represents the generalized factorial power with the step a, $a \in \mathbb{R}$, $m \in \mathbb{N}$. The following identities hold [16; Theorem 2.5]

$$\begin{cases} (B_n^{q,\alpha} e_0)(x) = 1, \quad (B_n^{q,\alpha} e_1)(x) = x, \\ (B_n^{q,\alpha} e_2)(x) = \dfrac{1}{1+\alpha}\left(x(x+\alpha) + \dfrac{x(1-x)}{[n]_q}\right), \ n \in \mathbb{N}, \ x \in [0,1]. \end{cases} \tag{6}$$

From the above, we easily deduce

$$\|B_n^{q,\alpha} e_2 - e_2\| = \frac{1}{4(1+\alpha)}\left(\alpha + \frac{1}{[n]_q}\right) \leq \alpha + \frac{1}{[n]_q}. \tag{7}$$

Also, we observe that the operators interpolate the approximated function f at the endpoints of the interval.

In order to transform the sequence $(B_n^{q,\alpha})_n$ into an approximation process, for each $n \in \mathbb{N}$, we replace the constants q and α by the numbers $q_n \in (0,1)$ and $\alpha_n \geq 0$, respectively.

Theorem 5 *Let the sequences $(q_n)_n$, $(\alpha_n)_n$ be given such that $0 < q_n < 1$ and $\alpha_n \geq 0$, $n \in \mathbb{N}$. Let the operators $B_n^{q_n,\alpha_n}$, $n \in \mathbb{N}$, be defined as in (4). If*

$$st - \lim_n q_n = 1 \quad \text{and} \quad st - \lim_n \alpha_n = 0, \tag{8}$$

then, for each $f \in C([0,1])$, one has

$$st - \lim_n \|B_n^{q_n,\alpha_n} f - f\| = 0. \tag{9}$$

Proof. Our assertion is implied by Theorem 1 Indeed, taking in view the identities (6), relation (7) and our hypothesis (8), all three requirements from (1) are fulfilled. The conclusion follows. □

Remark 5 *Set $S = \{10^k : \ k \in \mathbb{N}_0\}$ and consider the sequences*

$$q_n = \begin{cases} \dfrac{1}{n}, & n \in S, \\ 1 - \dfrac{1}{n}, & n \in \mathbb{N} \setminus S, \end{cases} \qquad \alpha_n = \begin{cases} \log n, & n \in S, \\ \dfrac{1}{n}, & n \in \mathbb{N} \setminus S. \end{cases}$$

Since relations (8) take place, statement (9) follows. On the other hand, $\lim_{n\to\infty} \|B_n^{q_n,\alpha_n} e_2 - e_2\|$ does not exist, see (7). In this way we exhibited an example for which the Popoviciu-Bohman-Korovkin theorem does not work but the statistical convergence works.

A sequence $(u_n)_n$ of real numbers satisfies the so-called *one-sided Tauberian condition* if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$

$$n\Delta u_n \geq -C, \tag{10}$$

where Δu_n denotes the *backward difference* $u_n - u_{n-1}$, with $u_{-1} = 0$.

Clearly, this condition is trivially satisfied if the involved sequence is non-decreasing. Fridy and Khan [10; Theorem 2.2] proved the following.

If condition (10) takes place, then

$$st - \lim_n u_n = L \quad \textit{implies} \quad \lim_n u_n = L.$$

On the basis of this result, we can state.

Theorem 6 *Let the sequences $(q_n)_n$, $(\alpha_n)_n$ be given such that $0 < q_n < 1$, $\alpha_n \geq 0$, $n \in \mathbb{N}$, and they satisfy condition (10). Let the operators $B_n^{q_n,\alpha_n}$, $n \in \mathbb{N}$, be defined as in (4). If (8) holds, then $(B_n^{q_n,\alpha_n})_n$ converges uniformly on $C([0,1])$ towards the identity operator.*

Now we are going to present the bivariate extension of q-Stancu operators. To achieve it, we use the method of parametric extensions of mentioned univariate operators. Both classical and new results concerning the tensor product operators can be found, e.g., in the well accomplished survey due to L. Beutel and H. Gonska [4].

Set $K = [0,1] \times [0,1]$ the unit square, and let $q(q_1, q_2)$ belong to the interior of K. We consider the parameter $\alpha(\alpha_1, \alpha_2) \in \mathbb{R}_+ \times \mathbb{R}_+$. For each $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$, we define the operator involving a cartesian product grid and acting on $C(K)$ as follows

$$(B_{n_1,n_2}^{\langle q,\alpha\rangle} f)(x_1, x_2) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} f(\lambda_{n_1,k_1,q_1}, \lambda_{n_2,k_2,q_2}) p_{n_1,k_1}^{q_1,\alpha_1}(x_1) p_{n_2,k_2}^{q_2,\alpha_2}(x_2), \tag{11}$$

$(x_1, x_2) \in K$, where $\lambda_{n_\nu,k_\nu,q_\nu = [k_\nu]_{q_\nu}/[n_\nu]_{q_\nu}}$ and $p_{n_\nu,k_\nu}^{q_\nu,\alpha_\nu}$, $0 \leq k_\nu \leq n_\nu$, are defined by (5), $\nu = 0, 1$.

For the particular case $\alpha = (0,0)$, $B_{n_1,n_2}^{\langle q,0\rangle}$ turns into bivariate q-Bernstein operators studied by D. Bărbosu [3].

Remark 6 *It's known that the tensor product of two univariate operators inherits many properties of its factors. For instance, the constructed operators are linear positive and interpolate the function f on the vertices of the unit square, this means*

$$(B_{n_1,n_2}^{\langle q,\alpha\rangle})(\tau_1, \tau_2) = f(\tau_1, \tau_2), \quad \tau_1, \tau_2 \in \{0, 1\}.$$

Moreover, taking into account (6), by a straightforward calculation, we deduce the images of the test functions $e_{i,j}$, $0 \leq i, j \leq 2$, under these operators.

Theorem 7 *Let $q(q_1, q_2) \in (0,1) \times (0,1)$ and $\alpha(\alpha_1, \alpha_2) \in \mathbb{R}_+ \times \mathbb{R}_+$. The operators $B_{n_1,n_2}^{\langle q,\alpha\rangle}$, $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$, defined by (11) verify the following identities*

$$B_{n_1,n_2}^{\langle q,\alpha\rangle} e_{i,j} = e_{i,j}, \quad (i,j) \in \{(0,0), (0,1), (1,0)\},$$

$$(B_{n_1,n_2}^{\langle q,\alpha\rangle} e_{2,0})(x_1, x_2) = (B_{n_1}^{q_1,\alpha_1} e_2)(x_1), \quad (B_{n_1,n_2}^{\langle q,\alpha\rangle} e_{0,2})(x_1, x_2) = (B_{n_2}^{q_2,\alpha_2} e_2)(x_2),$$

for each $(x_1, x_2) \in K$.

At this moment, following a similar course as in the one-dimensional case, we can establish conditions for ensuring the uniform or the statistical convergence of the sequence of operators to the identity operator. For each $n(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$, substituting α by $\alpha_n(\alpha_{1,n_1}, \alpha_{2,n_2})$, $\alpha_{\nu,n_\nu} \geq 0$, and q by $q_n(q_{1,n_1}, q_{2,n_2})$, $0 < q_{\nu,n_\nu} < 1$, where $\nu = 0, 1$, we get the following predictable result.

If $(\alpha_n)_n$ and $(q_n)_n$ are convergent (statistically convergent) to $(0,0)$ and $(1,1)$ respectively, then the sequence of operators is uniform convergent (statistically convergent) to the identity operator on the space $C(K)$.

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