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The Musical Mode from Mathematical Point of View

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Abstract

The history of the development of the concept of *mode* and *modality* in Music theory is more than 3000 years long. From the point of view of Combinatorics the musical mode is connected with the problem of decomposition of a positive natural number n in a sum of k numbers $(1 \leq k \leq n)$. The aim of the present paper is to establish a systematization of the *mode type* structures appearing in the so called *chromatic scale*. For that purpose we introduce a classification system and use Diophantine equations.

1 Introduction

The musical tones are produced by the vibration of different objects: strings, air columns in a pipe, bells, metallic or wooden plates, membranes etc. Their vibrations have different frequencies and we say that they produce different tones. The following fact is of major importance in the theory: Two tones with the ratio of their corresponding frequencies 2:1 have the same names in the usual system of appellation. In fact, they are different but they are similar from some aesthetic point of view. Theoretically as well practically it is said that the tone with the double frequency is higher than the other one by one *octave*.

It is necessary to make the following remark: Some notions from Music theory will appear below. To simplify the wording and spare long expressions and explications we shall not follow their exact definitions.

2 Scales

For the purposes of the *music aesthetics* it is necessary to determine the number and the frequency ratios of tones placed between two tones forming an octave. The first strict and complete investigations on this topic were done by Pythagoras. The concept of frequency appeared in the 17th century in a work of Isaac Beeckman (1588-1637). But the results of Pythagoras are calculated in ratios between the lengths of different parts of a string. The exact values of these ratios are not important for our investigations. We shall mention only that between the two end-tones of the octave there are six intermediate tones placed and two different ratios appear between the lengths of the respective parts of the string producing the new tones. The bigger ratio is called a *whole tone – wt* and the smaller one – *semitone – st*. The wt's are five and the st's are two. A tone sequence with the following ratios between the consecutive tones

wt – wt – st – wt – wt – wt – st

is called (*Pythagorean*) *diatonic scale*.

Many centuries later the development of the music practice and aesthetics introduced five more tones dividing the wt's into two st's. So, a new object appeared a sequence of 12 tones with equal consecutive ratios of the corresponding parts of string, namely 1 st. (In fact these st's were not equal.) This new object is called a *chromatic scale*.

3 Mode and Combinatorics

Looking on the Pythagorean scale we can say that it is constructed by seven objects – st's and wt's. Two of them are of one kind and five of another one. The number of variations of these objects is given by the well-known formula

$$\binom{7}{2} = \binom{7}{5} = \frac{7!}{5!2!} = 21$$

.

Let us think that the above sequence of wt's and st's are points on a circle. An illustration of the music practice in the last 4000 years or longer is the following: Choose arbitrarily one of the points and tour the circle. So we obtain seven sequences of wt's and st's. In each of them we shall never meet two neighbor st's. Moreover, the st's are always separated either by two or by three wt's. Roughly speaking, these seven sequences are called *modes*.

This last example is one reason to talk about the modes from the point of view of Combinatorics. The maximum possible combinations are 21 but only 7 are used in the practice. How to use or is it reasonable to use some other variations is a problem to be discussed by the musicians.

Since the wt consists of two st's, give the st the value 1 and the wt the value 2. Say that the value of the chromatic scale, calculated in st's, is 12. Following the same idea, the value of the diatonic scale is also 12. Comparing both cases we see that we have two examples of decomposition of the number 12 in a sum of natural numbers. That's one more reason to use Combinatorics to give some light on the abstract possibilities to compose tone sequences using the chromatic scale as a source.

The East Asian music uses the so called *pentatonic* scale. Inside the octave are placed four intermediate tones separated by

wt – wt+st – wt – wt – wt+st.

Or, the pentatonic is connected with following representation

2+3+2+2+3=12.

4 Subsets of the chromatic scale

Compare once again the chromatic and the diatonic scales. The chromatic consists of two tones forming an octave and 11 intermediate tones and we say that this is a 12-tone scale. The diatonic scale consists of the same two tones forming one octave and 6 intermediate tones; hence this is a 7-tone scale. Consequently the pentatonic is a 5-tone scale. Then the formula

$$\binom{11}{6} = 462$$

shows how much diatonic scales, in a larger sense of this notion, we can form, using the 12 tones of the chromatic scale. For the pentatonic this number is

$$\binom{11}{4} = 330$$

If we accept the idea that scales with another number of tones are also possible, the number of all scales would be

$$\sum_{i=0}^{11} \binom{11}{k} = 2^{11} = 2048$$

5 Intervals

In Music theory a pair of two different tones appearing together or consecutively is called *interval*. We need to distinguish the intervals numerically. For this purpose we introduce the notion *length* of an interval. The length is counted as the number of st's we can place between two different tones. So the diatonic scale contains intervals of lengths 1 and 2; the pentatonic contains intervals of lengths 2 and 3. And the chromatic scale contains only intervals of length 1.

The discussion of these three scales leads to the following ascertainment:

a) there are scales that contain different number of intervals;

b) there are scales that contain the same number of intervals but the intervals are not of the same lengths.

Then we can ask: *is it possible to compose scales by means of more than two intervals?*

And one more question: *a combination of intervals of which length could be used for the composition of different scales?*

6 The problem

Let us have an ordered string of tones, the first and the last tone forming an interval of length 12 (one octave). The intervals between the consecutive tones could be equal or different. (Here we avoid the notion scale.) Denote by ${}_kS(i_1, i_2, \ldots, i_k)$ the set of all such strings containing *k* different intervals with the lengths $i_1 < i_2 < \cdots < i_k < 12$. So, the chromatic scale as a tone string is an element of ${}_1S(1)$, the diatonic scale is an element of ${}_2S(1,2)$ and the pentatonic is an element of ${}_2S(2,3)$.

Our aim will be to list and investigate all sets ${}_kS(i_1, i_2, \ldots, i_k)$.

7 The Diophantine equation

Suppose we wish to know if it is possible to compose a tone string, determined by the intervals 2, 4 and 5. For this purpose we form the equation

$$2.x+4.y+5.z=12.$$

When such an equation allows only integer variables it is called *Diophantine equation*. In our case we are interested only in positive natural solutions. If the equation has a solution, *x* will be the number of intervals with length 2, *y* the number intervals with length 4 and *z* the number intervals with length 5. The quantity *x*+*y*+*z* is the number of different tones in the string we are looking for.

8 Systematization and solution

The introduction of the sets ${}_kS(i_1, i_2, \ldots, i_k)$ gives the possibility to describe systematically all strings of the above type, changing consecutively k and the length of the intervals. Then forming the respective Diophantine equation we shall look for its natural solutions.

8.1 Case *k*=1

We deal with the relatively simple set ${}_1S(i_1)$. To determine how much intervals with lenght i_1 we need, we have to solve in natural numbers the equation

$$i_1.x=12,\ i_1=1, 2, 3, \ldots, 12.$$

It is clear that solution exists only for i_1=1, 2, 3, 4, 6, 12 and x=12, 6, 4, 3, 2, 1 respectively.

So $\bigcup {}_1S(i_1)$ for all admissible values of i_1 contains **6** elements.

8.2 Case *k*=2

We have to solve the equation

$$i_1.x + i_2.y = 12$$

for all admissible values of i_1, i_2. It is clear that $i_1 + i_2 \le 12$. To check the admissibility of a pair we shall use the following conditions. A pair is admissible if either

12-(i_1+i_2)=0

or

12-(i_1+i_2)>0 and 12-(i_1+i_2) is divisible either by i_1 or by i_2.

8.2.1 Let $i_1 = 1, i_2 \in [2,11]$.

Since each number is divisible by 1, according to the above conditions, all pairs $(1, i_2), i_2 \in [2,11]$ are admissible. Then we have to solve the equations

$$x + i_2.y = 12, 2 \le i_2 \le 11.$$

- $i_2 = 2 \to {}_2S(1,2)$. The respective equation is x+2.y=12.

Then x=2.(6-y). And according to the conditions we have the possibilities y=1, 2, 3, 4, 5. So we obtain the following solutions:

(2, 5), (4, 4), (6, 3), (8, 2), (10, 1).

The strings corresponding to these solutions contain 7, 8, 9, 10 and 11 different tones respectively. The number of the different variations in each string is as follows

$$\binom{7}{2} = 21,\ \binom{8}{4} = 70,\ \binom{9}{3} = 84,\ \binom{10}{2} = 45,\ \binom{11}{1} = 11.$$

The number of elements of ${}_2S(1,2)$ is 21+70+84+56+11=**231**.

- $i_2 = 3 \to {}_2S(1,3)$. The respective equation is x+3.y=12.

Or x=3.(4-y) and we have necessarily y=1, 2, 3 and the solutions are

(3, 3), (6, 2), (9, 1).

So we have strings with 6, 8 and 10 tones respectively with the number of variations

$$\binom{6}{3} = 20,\ \binom{8}{2} = 28,\ \binom{10}{1} = 10.$$

The number of elements of ${}_2S(1,3)$ is 20+28+10=**58**.

- $i_2 = 4 \to {}_2S(1,4)$. The respective equation is $x+4.y=12$.

Now $x=4.(3-y)$ and $y=1, 2$ and the solutions are

(4, 2) and (8, 1)

determining strings with 6 and 9 tones respectively and variations

$$\binom{6}{2}=15,\ \binom{9}{1}=9.$$

The number of elements of ${}_2S(1,4)$ is 15+9=**24**.

- $i_2 = 5 \to {}_2S(1,5)$. The respective equation is $x+5.y=12$.

Write the equation in the form $y = 2+\frac{2-x}{5}$. It is equivalent to $\frac{2-x}{5} = \begin{cases}0\\5\end{cases}$. We obtain $x=2, 7$, two solutions

(2, 2), (7, 1)

and two strings with 4 and 8 tones and corresponding variations

$$\binom{4}{2}=6,\ \binom{8}{1}=8.$$

The number of elements of ${}_2S(1,5)$ is 6+8=**14**.

- $i_2 \in [6,11]$. Now $i_1 + i_2 > 6$ and the only solution is $(12-i_2,1)$. The corresponding string, one per case, consists of $12-i_2+1=13-i_2$ tones with number of variations

$$\binom{13-i_2}{1}=13-i_2.$$

Then we have $\sum_{i_2=6}^{11} 13-i_2 = \mathbf{27}$ variations in this group of cases.

Now we can summarize the variations of all cases when $i_1 = 1$ and the result is

231+58+24+14+27=**354**.

8.2.2 Let $i_1 = 2, i_2 \in [3,10]$.

There are two non-admissible pairs, namely (2, 7) and (2, 9).

- $i_2 = 3 \to {}_2S(2,3)$.

The equation $2.x+3.y=12$ written in the form $y = 4+\frac{2.x}{3}$ has the only solution (3, 2). Consequently we have only one string with 5 tones and number of variations of ${}_2S(2,3)$

$$\binom{5}{2}=\mathbf{10}.$$

- $i_2 = 4 \to {}_2S(2,4)$.

After a simple transformation of the equation $2.x+4.y=12$ we obtain $x=2.(3-y)$. The values $y=1, 2$ lead to the solutions (2, 2), (4, 1) and two strings with 4 and 6 tones and corresponding numbers of variations

$$\binom{4}{2} = 6 \text{ and } \binom{5}{1} = 5.$$

The number of elements of ${}_2S(2,4)$ is 6+5=**11**.

- $i_2 = 5 \rightarrow {}_2S(2,5)$; $12-(i_1+i_2) = 12-(2+5) = 5$.

This calculation shows that we have only one solution – (1, 2). The string has three tones and the number of elements of ${}_2S(2,5)$ is **3**.

- $i_2 = 6 \rightarrow {}_2S(2,6)$.

After transformation of the standard equation we obtain $y = 1 - \frac{x}{3}$. The solution is (3, 1) – four tones string.

The number of elements of ${}_2S(2,6)$ is **4**.

- (2, 7) and (2, 9) are not admissible.
- $i_2 = 8 \rightarrow {}_2S(2,8)$ and $i_2 = 10 \rightarrow {}_2S(2,10)$.

Both cases are quite clear and we have

${}_2S(2,8)$ - **3** elements, ${}_2S(2,10)$ - **2** elements.

The variations of all cases when $i_1 = 2$ are

10+11+3+4+3+2=**33**.

8.2.3 Let $i_1 = 3, i_2 \in [4,9]$.

- $(3,i_2), i_2 = 4,5,7,8$ are not-admissible pairs.
- $i_2 = 6 \rightarrow {}_2S(3,6)$ and $i_2 = 9 \rightarrow {}_2S(3,9)$.

The variations of all cases when $i_1 = 3$ are

2+3=**5**.

8.2.4 Let $i_1 = 4, i_2 \in [5,8]$.

- $i_2 = 8 \rightarrow {}_2S(4,8)$ is the only admissible case.

The variations of all cases when $i_1 = 4$ are **2**.

8.2.5 Let $i_1 = 5, i_2 \in [6,7]$.

- $i_2 = 7 \rightarrow {}_2S(5,7)$ is the only admissible case.

The variations of all cases when $i_1 = 5$ are **2**.

8.2.6 Let $i_1 = 6, i_2 = 6$.

${}_2S(6,6) \equiv {}_1S(2)$. It is already counted.

Then we can summarize.

The total of variations for the case *k*=2 is

354+33+5+2+2=**<u>396</u>**.

8.3 Case *k*=3

We have a similar condition for the admissibility of a triple (i_1, i_2, i_3). The general form of the Diophantine equation is

$$i_1.x + i_2.y + i_3.z = 12.$$

The solutions will give tone strings with *x*+*y*+*z* tones. The number of variations in each string is given by the formula

$$\frac{(x+y+z)!}{x!y!z!}.$$

8.3.1 Let $i_1 = 1$.

- $i_2 = 2, i_3 \in [3, 9]$.
  - $i_3 = 3 \rightarrow {}_3S(1, 2, 3)$. We have to solve the equation *x*+2.*y*+3.*z*=12. Rewrite it in the

form

$$x + 2.y = 3.(4 - z)$$

having in mind that $x + 2.y \geq 3$. Then we have *z*=1, 2, 3. Putting *z*=1 we obtain a new equation *x*+2.*y*=9. In item **2.** we solved many similar equations therefore we shall give directly the solutions.

(1, 4, 1), (3, 3, 1), (5, 2, 1), (7, 1, 1)

We obtained four strings with 6, 7, 8 and 9 tones respectively. The variations in each string are

$\frac{6!}{1!4!1!} = 30$, $\frac{7!}{3!3!1!} = 140$, $\frac{8!}{5!2!1!} = 168$, $\frac{9!}{7!7!1!} = 72$, or in total 30+140+168+72=410.

For *z*=2, 3 we obtain three mire solutions

(2, 2, 2), (4, 1, 2), (1, 1, 3)

with variations

$\frac{6!}{2!2!2!} = 90$, $\frac{7!}{4!1!2!} = 105$, $\frac{5!}{1!1!3!} = 20$ with the total 90+105+20=215.

The number of variations in ${}_3S(1, 2, 3)$ is 410+215=**625**.

- $i_3 = 4 \rightarrow {}_3S(1, 2, 4)$. We have to solve the equation *x*+2.*y*+4.*z*=12. Rewrite it in the

form

$$x + 2.y = 4.(3 - z).$$

We repeat the above procedure and putting *z*=1, 2 we obtain the solutions

(2, 1, 2), (2, 3, 1), (4, 2, 1), (6, 1, 1)

with variations

$$\frac{5!}{2!\,1!\,2!} = 30,\ \frac{6!}{2!\,3!\,1!} = 60,\ \frac{7!}{4!\,2!\,1!} = 105,\ \frac{8!}{6!\,1!\,1!} = 56.$$

The number of variations in ${}_3S(1, 2, 4)$ is 30+60+105+56=**251**.

- $i_3 = 5 \rightarrow {}_3S(1, 2, 5)$. Since 1+2+5=8 it is clear that *z*=1. The new equation *x*+2.*y*=7

gives three solutions

(1, 3, 1), (3, 2, 1), (5, 1, 1)

with variations

$$\frac{5!}{1!\,3!\,1!} = 20,\ \frac{6!}{3!\,2!\,1!} = 60,\ \frac{7!}{5!\,1!\,1!} = 42.$$

The number of variations in ${}_3S(1, 2, 5)$ is 20+60+42=122.

- $i_3 = 6,7,8,9 \rightarrow {}_3S(1, 2, i_3)$. Since $1+2+i_3 \geq 9$ z=1. The solutions are
  - $i_3 = 6 \rightarrow$ (2, 2, 1), (4, 1, 1);
  - $i_3 = 7 \rightarrow$ (1, 2, 1), (3, 1, 1);
  - $i_3 = 8 \rightarrow$ (2, 1, 1);
  - $i_3 = 9 \rightarrow$ (1, 1, 1).

Here are the corresponding variations

$$\frac{5!}{2!\,2!\,1!} = 30, \quad \frac{6!}{4!\,1!\,1!} = 30, \quad \frac{4!}{1!\,2!\,1!} = 12, \quad \frac{5!}{3!\,1!\,1!} = 20, \quad \frac{4!}{2!\,1!\,1!} = 12, \quad \frac{3!}{1!\,1!\,1!} = 6.$$

The number of variations in $\bigcup_{i_3=3}^{9} {}_3S(1, 2, i_3)$ is 625+251+122+60+32+12+6=**1108**.

- $i_2 = 3, i_3 \in [4,8]$.
  - $i_3 = 4 \rightarrow {}_3S(1, 3, 4)$. The transformed equation is $x + 3.y = 4.(3 - z) \geq 4$. Hence

z=1,2. We have three solutions

(2, 2, 1), (5, 1, 1), (1, 1, 2).

The variations:

$$\frac{5!}{2!\,2!\,1!} = 30, \quad \frac{7!}{5!1!1!} = 42, \quad \frac{4!}{1!1!2!} = 12$$ Total in ${}_3S(1, 3, 4)$ - 30+42+12=84.

- $i_3 = 5 \rightarrow {}_3S(1, 3, 5)$. Clearly z=1 and the equation x+3.y=7 gives the solutions

(1, 2, 1), (4, 1, 1).

The variations:

$$\frac{4!}{1!2!1!} = 12, \quad \frac{6!}{4!1!1!} = 30.$$ Total in ${}_3S(1, 3, 5)$ – 12+30=42.

- $i_3 = 6, 7, 8 \rightarrow {}_3S(1,3,i_3)$.

$$1 + 3 + i_3 \geq 10 \Rightarrow z = 1$$

$$x + 3.y = 12 - i_3 \geq 6 \Rightarrow y = 1$$

The solutions are

- $i_3 = 6 \rightarrow$ (3, 1, 1),
- $i_3 = 7 \rightarrow$ (2, 1, 1),
- $i_3 = 8 \rightarrow$ (1, 1, 1)

with variations

$$\frac{5!}{3!\,1!\,1!} = 20\,,\ \frac{4!}{2!\,1!\,1!} = 12\,,\ \frac{3!}{1!1!1!} = 6\,.$$

The number of variations in $\bigcup_{i_3=4}^{8} {}_3S(1,3,i_3)$ is 84+42+20+12+6=**164**.

- $i_2 = 4, i_3 = 5,6,7$. As earlier we conclude that $y=z=1$ and we have one solution per case (3, 1, 1), (2, 1, 1), (1, 1, 1).

The variations

$\frac{5!}{3!\,1!\,1!} = 20\,,\ \frac{4!}{2!\,1!\,1!} = 12\,,\ \frac{3!}{1!1!1!} = 6$ have the total 20+12+6=**38**.

- $i_2 = 5, i_3 = 6 \rightarrow {}_3S(1,5,6)$. There is only one solution (1, 1, 1) with **6** variations.

The number of variations in $\bigcup {}_3S(1,i_2,i_3)$ for all admissible triples $(1,i_2\,i_3)$ is 1108+164+38+6=**1316**.

8.3.2 Let $i_1 = 2$.

- $i_2 = 3$. The triple (2, 3, 6) is not-admissible.
  - $i_3 = 4$. Solution: (1, 2, 1). Variations: $\frac{4!}{1!\,2!\,1!} = 12$.
  - $i_3 = 5$. Solution: (2, 1, 1). Variations: $\frac{4!}{2!\,1!\,1!} = 12$.
  - $i_3 = 7$. Solution: (1, 1, 1). Variations: $\frac{3!}{1!\,1!\,1!} = 6$.
- $i_2 = 4$. The triple (2, 4, 5) is not-admissible.
  - $i_3 = 6$. Solution: (1, 1, 1). Variations: $\frac{3!}{1!\,1!\,1!} = 6$.

The number of variations in $\bigcup {}_3S(2,i_2,i_3)$ for all admissible triples $(2,i_2\,i_3)$ is 12+12+6+6=**36**.

8.3.3 Let $i_1 = 3$.

There is only one admissible triple (3, 4, 5). Solution: (1, 1, 1). Variations: $\frac{3!}{1!\,1!\,1!} = 6$. And the total is **6**.

The total of variations for the case k=3 is

1216+36+6=**<u>1258</u>**.

8.4 Case *k*=4.

The corresponding equation is $i_1.x + i_2.y + i_3.u + i_4.v = 12$. From 1+2+3+4=10 we see that $u = v = 1$.

- $i_1 = 1, i_2 = 2, i_3 = 3, i_4 \in [4,5,6]$.

- $i_4 = 4 \rightarrow {}_4S(1,2,3,4)$. The equation $x+2.y=5$ gives the solutions

(1, 2, 1, 1), (3, 1, 1, 1)

with variations

$$\frac{5!}{1!\,2!\,1!\,1!} = 60, \quad \frac{6!}{3!\,1!\,1!\,1!} = 120.$$

- $i_4 = 5,6 \rightarrow {}_4S(1,2,3,5),\ {}_4S(1,2,3,6)$. Since $1+2+3+i_4 \geq 11$, we have $y=u=v=1$.

Solutions: (2, 1, 1, 1), (1, 1, 1, 1). Variations:

$$\frac{5!}{2!\,1!\,1!\,1!} = 60, \quad \frac{4!}{1!\,1!\,1!\,1!} = 24 \cdot$$

- $i_1 = 1, i_2 = 2, i_3 = 4, i_4 = 5 \rightarrow {}_4S(1,2,4,5)$. There is only one solution (1, 1, 1, 1) with

$$\frac{4!}{1!\,1!\,1!\,1!} = 24$$

variations.

The total of variations for the case k=4 is

60+120+60+24+24=**288**.

At the end of this investigation we can summarize the results for the four cases:

6+396+1358+288=2048.

So we enumerated all tone strings.

9 Examples.

Some examples was given the introduction part. We shall repeat them we shall add some other practical cases.

1. The well-known natural major and minor scales are contained in ${}_2S(1,2)$ and especially in the solution (2, 5).

2. The so called *melodic* major and minor are also in ${}_2S(1,2)$ by the same solution. But all the four scales are different variations of the solution (2, 5).

3. Another variety of the major/minor scale is the so called *harmonic* scale. It is contained in ${}_3S(1,2,3)$ and is variation of the solution (3, 3, 1).

4. The pentatonic is contained in ${}_2S(2,3)$ and given by the solution (3, 2).

5. ${}_1S(1)$ is the chromatic scale.

6. ${}_1S(6)$ is the so called *whole tone* scale. It was used in the first half of the 20th century by Rimski-Korsakov, Debussy, Ravel and other composers.

7. The music 0f the 20th and 21st uses *octatonic*. This is variation of the solution (4, 4) in ${}_2S(1,2)$.

8. In the 11th-12th century the composers used *hexatonic*. The names of the tones come from a very popular hymn dedicated to Saint John composed by Guido d'Arezzo. The hexatonic used comes from ${}_3S(1,2,3)$, solution (1, 4, 1).

After a careful study of different musical compositions one could find many other cases, contained among the 2084 tone strings described above.

References

[1] Kenneth R. Rumery, *Octatonic Scales*, http://jan.ucc.nau.edu/~krr2/ct_octindex.html, 1996.
[2] Klaus Lang,*Auf Wohlklangswellen durh der Toene Meer*, Graz, 1999.
[3] Kyle Gann, *An Introduction to Historical Tunings*, http://www.kylegann.com/histune.html, 1997.
[4] L.Krasinskaya, V.Utkin, *Elementary Music Theory* (in Russian), Moscow, 1983.
[5] Peter A. Frazer, *The Development of Musical Tuning Systems*, http://www.midicode.com/tunings/, 2001.
[6] Stefan Goeller, *Introduction to Musical Scales*, University of Warwick, 2001.
[7] V.Serpinskii, *On Entire Numbers Solutions of Equations* (in Russian), Moscow, 1961

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