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Center of a Set of Points in Three-dimensional Space Using Triangular Metric

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Abstract

The triangular distance between two points A(xa, ya, za) and B(xb, yb, zb) in three-dimensional space is defined as follows:

$$R_T = |x_a - x_b| + |y_a - y_b| + |z_a - z_b|$$

In the present paper is discussed and solved the following problem: given a set of points to be found such a point , denoted center, that minimizes:

$$\max R_T(M, A_i) \tag{1}$$

Based on the results in [1], where the two-dimensional case is analyzed, the authors prove that (1) has not a single solution and in the general case it is a two-dimensional convex simplex which is an intersection of n octahedra. The graphical solution is obtained by a developed computer program.

1 Introduction

The problem of finding the center of a set of points is an important problem in a number of mathematical models in transport, computer networks and communications and other engineering networks. The location of the points, on the vertices of a given graph, on the nodes of rectangular two-dimensional or three-dimensional network, is important [2]. For this reason a given metric is suitable in one case and different metric is suitable in other case. In the present paper triangular (Manhattan) metric is used that is suitable for finding the center of points in rectangular networks. The application of the latter metric helps to avoid the complex mathematical methods for nonlinear optimisation in solving the problem just stated which are not able to give all the possible geometrical solutions.

2 Formulation of the Problem

Triangular distance between two points A(xa, ya, za) and B(xb, yb, zb) in three-dimensional space is defined as follows:

$$R_T(A, B) = |x_a - x_b| + |y_a - y_b| + |z_a - z_b|$$

Given a set P of points $A_i(x_i, y_i, z_i)$ $i = 1, 2, ..., n$, the center of P will be such a point $M(x, y, z)$, that minimizes

$$\max R_T(M, A_i) \quad i = 1, 2, ..., n \tag{1}$$

In [1] this problem is solved for the two-dimensional case. There it is defined as follows:

$$\min \max \rho_i (|x - x_i| + |y - y_i|) \tag{2}$$

where $\rho_i \geq 0$ are weights.

The following problem proves the equivalence of (2):

$$\min r \tag{3}$$

for:

$$|x - x_i| + |y - y_i| \leq \frac{r}{\rho_i}, \quad i = 1, 2, ..., n \tag{4}$$

3 The Two-dimensional Case

The algorithm for the implementation of (3) and (4), described in [1], is applied in the three-dimensional case.

The limitation (4) defines a set of points (x, y) whose triangular distance to (x_i, y_i) is not greater than $\frac{r}{\rho_i}$. It is not difficult to estimate that this set is the interior of a square whose diagonals intersect at point (x_i, y_i) and is rotated about the coordinate system by 45°. And the square is a set of geometric points at an equal triangular distance from the point (x_i, y_i) (Figure 1).

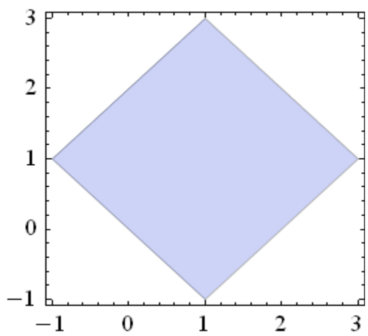

Figure 1

Considering the above, the problem is reduced to finding the smallest value of r, for which the intersection of the squares $|x - x_i| + |y - y_i| \leq \frac{r}{\rho_i}, \quad i = 1, 2, ..., n$ is not empty (Figure 2).

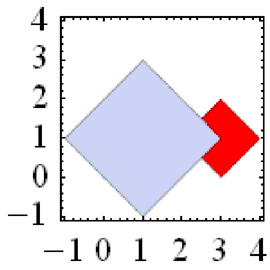

Figure 2

The basic steps of the algorithm for finding $\min r$ are:

- Coordinate system Oxy is rotated by 45°. In the new coordinate system Ox'y' the areas (4) are defined by the inequalities:

$$x_i' - \frac{\sqrt{2}r}{2\rho_i} \le x' \le x_i' - \frac{\sqrt{2}r}{2\rho_i}; \qquad y_i' - \frac{\sqrt{2}r}{2\rho_i} \le y' \le y_i' - \frac{\sqrt{2}r}{2\rho_i} \quad i = 1, 2, \ldots\ldots, n \qquad (5)$$

- What is this min r, for which the intervals (5) have non-empty intersection. Following [1] min r =max(r_1, r_2), where

$$r_1 = \max \eta_{ij}; \qquad r_2 = \max \delta_{ij}, \; i = 1, 2, \ldots, n-1, \quad j = i+1, \ldots, n$$

$$\eta_{ij} = \frac{\rho_i \rho_j}{\rho_i + \rho_j}\sqrt{2}\left|x_i' - x_j'\right|; \quad \delta_{ij} = \frac{\rho_i \rho_j}{\rho_i + \rho_j}\sqrt{2}\left|y_i' - y_j'\right|$$

- When r_1=r_2=r the solution is one and only one and this is the point M with coordinates

$$x_M' = x_{i_1}' + \frac{\sqrt{2}r}{2\rho_{i_1}}; \quad y_M' = y_{i_2}' + \frac{\sqrt{2}r}{2\rho_{i_2}}$$

- When r_1>r_2 (r_2>r_1) the solution is a line segment bounded by the two end points

$$\left(x_M' = x_{i_1}' + \frac{\sqrt{2}r_1}{2\rho_{i_1}}, y_{M_1}' = \max\left(y_i' - \frac{\sqrt{2}r_1}{2\rho_i}\right)\right), \quad \left(x_M' = x_{i_1}' + \frac{\sqrt{2}r_1}{2\rho_{i_1}}, y_{M_1}' = \min\left(y_i' - \frac{\sqrt{2}r_1}{2\rho_i}\right)\right)$$

then the transformation Ox'y' $\rightarrow$ Oxy is repeated

4 The Three-dimensional Case

In the three-dimensional case problem (1) can be formulated as follows:

$$\min R \qquad (6)$$

for:

$$|x - x_i| + |y - y_i| + |z - z_i| \le R; i = 1, 2, ..., n \tag{7}$$

Each of the limitations (7) is a regular octaedron, which image is shown in Figure 3.

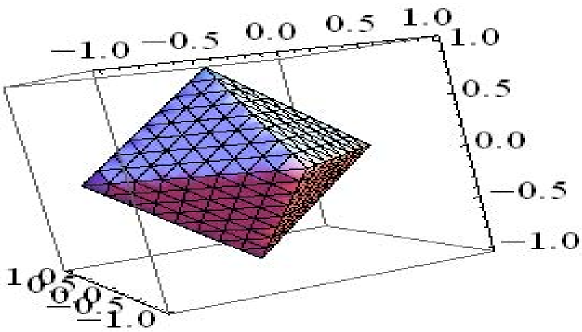

Figure 3

(6)-(7) are equivalent to

$$min\ R \tag{8}$$

$$|x - x_i| + |y - y_i| \le R - d_i(z); \quad d_i(z) = |z - z_i|; \qquad i = 1, 2, ..., n \tag{9}$$

If **z** is fixed, the above problem is two-dimensional.
The coordinate system Oxy is rotated by 45° and the problem is reduced to finding the minR(z), where the intersection of the squares

$$\begin{aligned} x_i' - k(R(z) - d_i(z)) \le x' \le x_i' + k(R(z) - d_i(z)) \\ y_i' - k(R(z) - d_i(z)) \le y' \le y_i' + k(R(z) - d_i(z)) \end{aligned} \tag{10}$$

is not empty.($k = \frac{\sqrt{2}}{2}$).

4.1 Three-dimensional algorithm

1. For each pair of points x_i' u x_j' is seeked $\alpha_{ij}(z) = \min(R(z))$, for which the intervals with centres x_i' u x_j' have a common point. Considering (10), it is easy to prove that:

$$Rx_{ij}(z) = k|x_i' - x_j'| + 0.5|d_i(z) + d_j(z)\ | \tag{11}$$

The same result is obtained on the axis Oy:

$$Ry_{ij}(z) = k|y_i' - y_j'| + 0.5|d_i(z) + d_j(z)\ | \tag{12}$$

These functions look like as shown in Figure 4.

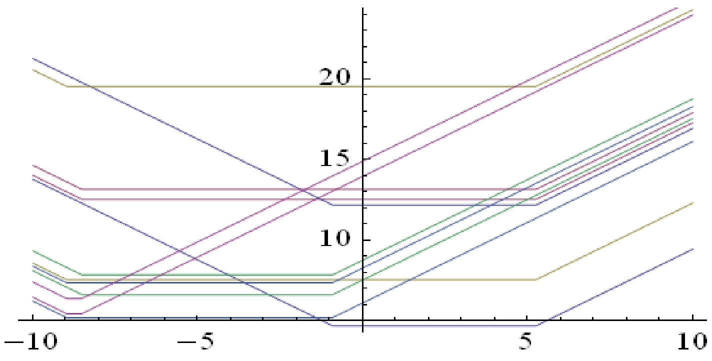

Figure 4

2. Considering the two sets of functions (11), (12), is seeked

$$R(z) = \max\{\max_{i,j} Rx_{ij}(z), \max_{i,j} Ry_{ij}(z)\} \tag{13}$$

It is obvious that $R(z)$ is a line segment, parallel to the axis Oz, and $z_{min} \le z \le z_{max}$

3. Defining z_{min} and z_{max}

   The two end points of the interval $z_{min} \le z \le z_{max}$ are defined as the intersection of $R(z)$ with the functions (11), (12).

4. From (10) are found the intervals (as function of z) within which x’ and y’ vary.

   The boundaries of the area of variation of x’ and y’ for a particular instance are shown in Figure 5.

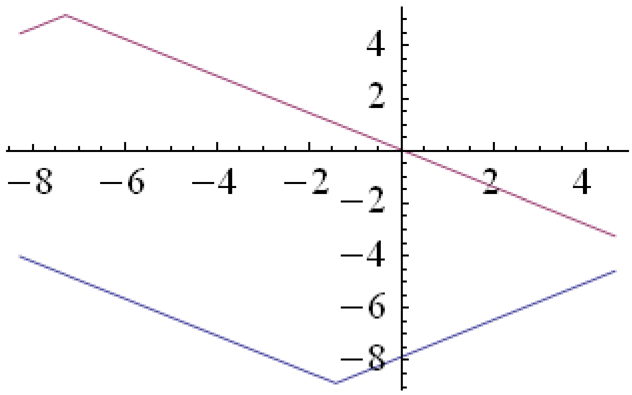

Figure 5

5. The center of the set of given points is drawn, which in the general case is a two-dimensional simplex in the three-dimensional space.

   In Figure 6 is given an instance of computed center of 10 points, generated at random.

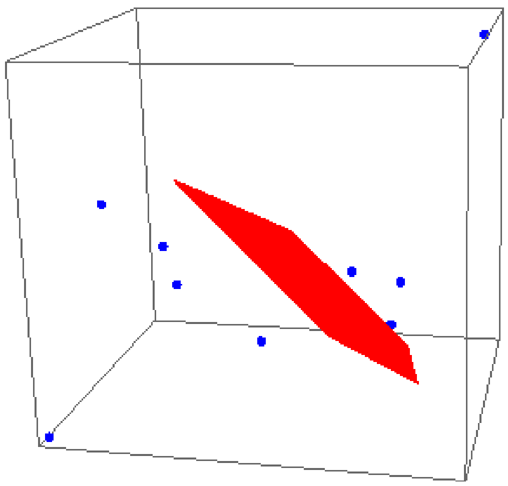

Figure 6

5 Conclusion

A computer program is developed that implements the proposed algorithm. The instances presented are generated by this program. For this purpose the program system Mathematica 7.0 with its rich graphical functionality is used.

References

[1] P. Lalov, T. Vasileva, *Extremal Problems of Triangular Metric, University Annual Applied Mathematics*, Volume 21, Sofia, 1985.

[2] Thomas L. Saaty, *Optimization in Integer and Related Extremal Problems*, McGraw Hill, New York, 1970.

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